23 Tutorial 3: Orthogonal decomposition, duals and bases

Learn to prove the following theorems, accurately, efficiently, using quality proof machine, without having to refer to notes. The first step of this process is to work through each and put the reason why each line appears where it appears. The possible reasons are:

(a) (Proof type II) Assume the ifs
(b) (Proof type II) To show the thens
(c) (Rewriting) This is the definition of _______.
(d) (Proof type III) To show something exists, construct it.
(e) (Proof type III) To show the construction is valid.
(f) (Proof type I) Compute the left hand side.
(g) (Proof type I) Compute the right hand side.

Practice each proof so that you can do it efficiently without referring to notes.

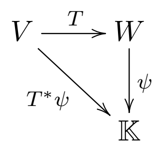
23.0.1 Duals and adjoints

Let V be a normed vector space. The space of *bounded linear functionals on* V, or the *dual of* V, is

$$V^* = B(V,\mathbb{R}) = \{\text{bounded linear transformations } \varphi\colon V\to\mathbb{R}\}.$$

Let $(V,\|\ \|_V)$ and $(W,\|\ \|_W)$ be normed vector spaces. Let $T\colon V\to W$ be a linear operator. The *adjoint of* T is the linear transformation

$$T^*\colon W^*\to V^* \quad \text{given by} \quad (T^*\varphi)(v) = \varphi(T(v)).$$



Proposition 23.1. *Let H be a Hilbert space. Then*

$$\begin{array}{rccc} \Psi\colon & H & \longrightarrow & H^* \\ & x & \longmapsto & \Psi_x \end{array} \quad \textit{where} \quad \begin{array}{rccc} \Psi_x\colon & H & \to & \mathbb{K} \\ & h & \to & \langle h, x\rangle \end{array}$$

is a skew-linear bijective isometry and $\|\Psi\| = 1$.

The dual H^* does not have a natural inner product so it is not naturally a Hilbert space until it is identified with H. The proof of Proposition 23.1 uses the following Theorem from Tutorial 2.

Theorem 23.2. *(Orthogonal decomposition) Let V be a Hilbert space. Let W be a subset of V.*

(a) $W^\perp$ is a closed subspace of V.

(b) W is a closed subspace of V if and only if $V = W \oplus W^\perp$.

23.0.2 Bases

Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. Let V be a $\mathbb{K}$-vector space.
A *basis of* V is a subset $B \subseteq V$ such that

(a) $\mathbb{K}\text{-span}(B) = V$,

(b) B is linearly independent,

where

$$\mathbb{K}\text{-span}(B) = \{a_1b_1 + \cdots + a_\ell b_\ell \mid \ell \in \mathbb{Z}_{>0},\ b_1, \ldots, b_\ell \in B,\ a_1, \ldots, a_\ell \in \mathbb{K}\}$$

and B is *linearly independent* if B satisfies

$$\text{if } \ell \in \mathbb{Z}_{>0} \text{ and } b_1, \ldots, b_\ell \in B \text{ and } a_1, \ldots, a_\ell \in \mathbb{K}, \text{ and}$$

$$a_1b_1 + \cdots a_\ell b_\ell = 0 \quad \text{then} \quad a_1 = 0,\ a_2 = 0,\ \ldots,\ a_\ell = 0.$$

Let V be a topological $\mathbb{K}$-vector space. A *topological basis of* V is a subset $B \subseteq V$ such that

(a) $\overline{\mathbb{K}\text{-span}(B)} = V$,

(b) B is linearly independent,

Proposition 23.3. *Let $\mathbb{K}$ be $\mathbb{R}$ of $\mathbb{C}$ and let $(V, \|\ \|)$ be a normed $\mathbb{K}$-vector space. Then V has a countable dense set C if and only if V has a sequence $B = (b_1, b_2, \ldots)$ with $\overline{\mathbb{K}\text{-span}(B)} = V$.*

24 Tutorial 3: Solutions

24.1 Orthogonal decomposition

Theorem 24.1. *Let V be a Hilbert space. Let W be a subset of V.*

(a) $W^\perp$ is a closed subspace of V.

(b) W is a closed subspace of V if and only if $V = W \oplus W^\perp$.

Proof. (a) Let

$$\begin{array}{rccc} \Phi\colon & V & \to & W^* \\ & v & \mapsto & \Phi_v \end{array} \quad \text{where} \quad \begin{array}{rccc} \Phi_v\colon & W & \to & \mathbb{K} \\ & w & \mapsto & \langle v, w\rangle \end{array}$$

Then

$$W^\perp = \{v \in V \mid \text{if } w \in W \text{ then } \langle v, w\rangle = 0\} = \{v \in V \mid \Phi_v = 0\} = \ker\Phi = \Phi^{-1}(\{0\}).$$

Since $\{0\}$ is closed in W^* and Φ is continuous then $W^\perp = \Phi^{-1}(\{0\})$ is closed.
(b) Assume that W is a closed subspace of V.
To show: (ba) $H = \overline{W} + \overline{W}^\perp$.
(bb) $\overline{W} \cap \overline{W}^\perp = 0$.

(ba) If $x \in H$ then $x = P(x) + (x - P(x)) \in \overline{W} + \overline{W}^\perp$.
So $H = \overline{W} + \overline{W}^\perp$.
(bb) If $y \in \overline{W} \cap \overline{W}^\perp$ then $\langle y, y\rangle = 0$, forcing that $y = 0$.
So $\overline{W} \cap \overline{W}^\perp = 0$.

□

24.2 The dual of a Hilbert space

Proposition 24.2. (see [Bre, Theorem 5.7]) *Let H be a Hilbert space. Then*

$$\begin{array}{rccc} \Psi\colon & H & \longrightarrow & H^* \\ & x & \longmapsto & \Psi_x \end{array} \quad \textit{where} \quad \begin{array}{rccc} \Psi_x\colon & H & \to & \mathbb{K} \\ & h & \to & \langle h, x\rangle \end{array}$$

is a skew-linear bijective isometry and $\|\Psi\| = 1$.

Proof.

To show: (a) Ψ is skew-linear.
(b) Ψ is an isometry.
(c) $\|\Psi\| = 1$.
(d) Ψ is injective.
(e) Ψ is surjective.

(a) To show: (aa) If $a, b \in H$ then $\Psi_{a+b} = \Psi_a + \Psi_b$.
(ab) If $a \in H$ and $c \in \mathbb{K}$ then $\Psi_{ca} = \bar{c}\Psi_a$.

(aa) Assume $a, b \in H$.
To show: $\Psi_{a+b} = \Psi_a + \Psi_b$.
To show: If $h \in H$ then $\Psi_{a+b}(h) = \Psi_a(h) + \Psi_b(h)$.

Assume $h \in H$.
To show: $\Psi_{a+b}(h) = \Psi_a(h) + \Psi_b(h)$.

$$\Psi_{a+b}(h) = \langle h, a+b\rangle = \langle h, a\rangle + \langle h, b\rangle = \Psi_a(h) + \Psi_b(h).$$

(ab) Assume $a \in H$ and $c \in \mathbb{K}$.
To show: $\Psi_{ca} = \bar{c}\Psi_a$.
To show: If $h \in H$ then $\Psi_{ca}(h) = \bar{c}\Psi_a(h)$.
Assume $h \in H$.
To show: $\Psi_{ca}(h) = \bar{c}\Psi_a(h)$.

$$\Psi_{ca}(h) = \langle h, ca\rangle = \bar{c}\langle h, a\rangle = \bar{c}\Psi_a(h).$$

(b) Let $h \in H$. By Cauchy-Schwarz,

$$|\Psi_x(h)| = |\langle h, x\rangle| \leq \|h\| \cdot \|x\|.$$

So $\|\Psi_x\| \leq \|x\|$.
Since

$$\|\Psi_x(x)\| = |\langle x, x\rangle| = \|x\|^2 = \|x\| \cdot \|x\|,$$

then $\|\Psi_x\| \geq \|x\|$.
So $\|\Psi_x\| = \|x\|$.
So Ψ is an isometry.

(c) To show: $\|\Psi\| = 1$.
Using that $\|\Psi_x\| = \|x\|$ from part (b),

$$\|\Psi\| = \sup\left\{ \frac{\|\Psi_x\|}{\|x\|} \mid x \in H, x \neq 0 \right\} = \sup\{1\} = 1.$$

(d) To show: Ψ is injective.
To show: If $a, b \in H$ and $\Psi_a = \Psi_b$ then $a = b$.
Assume $a, b \in H$ and $\Psi_a = \Psi_b$.
To show: $a = b$.
To show: $\|a - b\| = 0$.

$$\|a - b\| = \|\Psi_{a-b}\| = \|\Psi_a - \Psi_b\| = \|0\| = 0.$$

So $a = b$.
So $\Psi$is injective.

(e) To show: Ψ is surjective.
To show: If $\varphi \in H^*$ then there exists $a \in H$ such that $\varphi = \Psi_a$.
Assume $\varphi \in H^*$.
To show: There exists $a \in H$ such that $\varphi = \Psi_a$.
Case 1: $\varphi = 0$. Then $\varphi = \Psi_0$.
Case 2: $\varphi \neq 0$.
Since φ is bounded then φ is continuous.
Since $\{0\}$ is closed in $\mathbb{K}$ then $\ker \varphi = \varphi^{-1}(\{0\})$ is closed in H.

By the orthogonal decomposition theorem (Theorem 23.2), $H = \ker\varphi \oplus (\ker\varphi)^\perp$.
Let

$$b \in (\ker\varphi)^\perp \quad \text{with } b \neq 0 \qquad \text{and let} \qquad a = \frac{\overline{\varphi(b)}}{\|b\|^2} b.$$

To show: If $h \in H$ then $\varphi(h) = \Psi_a(h)$.
Assume $h \in H$,

$$h = \left(h - \frac{\varphi(h)}{\varphi(a)} a\right) + \frac{\varphi(h)}{\varphi(a)} a, \qquad \text{with} \quad \left(h - \frac{\varphi(h)}{\varphi(a)} a\right) \in \ker\varphi.$$

To show: $\varphi(h) = \Psi_a(h)$.
Since

$$\langle a, a\rangle = \left\langle \frac{\overline{\varphi(b)}}{\|b\|^2} b, \frac{\overline{\varphi(b)}}{\|b\|^2} b \right\rangle = \frac{\overline{\varphi(b)}}{\|b\|^2} \frac{\varphi(b)}{\|b\|^2} \langle b, b\rangle = \frac{\overline{\varphi(b)}}{\|b\|^2} \varphi(b) = \varphi(a).$$

and since $a \in (\ker\varphi)^\perp$ then

$$\Psi_a(h) = \langle h, a\rangle = \left\langle \left(h - \frac{\varphi(h)}{\varphi(a)} a\right) + \frac{\varphi(h)}{\varphi(a)} a, a \right\rangle = 0 + \frac{\varphi(h)}{\varphi(a)} \langle a, a\rangle = \varphi(h),$$

So $\Psi_a = \varphi$.
So Ψ is surjective.

□

24.3 Separable Hilbert spaces have countable topological bases

Proposition 24.3. *Let $\mathbb{K}$ be $\mathbb{R}$ of $\mathbb{C}$ and let $(V, \| \ \|)$ be a normed $\mathbb{K}$-vector space. Then V has a countable dense set C if and only if V has a sequence $B = (b_1, b_2, \ldots)$ with $\overline{\mathbb{K}\text{-span}(B)} = V$.*

Proof. $\Rightarrow$: Assume C is a countable dense subset of V.
To show: There is a countable subset $B \subseteq V$ with $\overline{\mathbb{K}\text{-span}(B)} = V$.
Let $B = C$.
Since $C \subseteq \mathbb{K}\text{-span}(C) = \mathbb{K}\text{-span}(B)$ then $V = \overline{C} = \overline{\mathbb{K}\text{-span}(B)}$.
So $V = \overline{\mathbb{K}\text{-span}(B)}$.
$\Leftarrow$: Assume V has a countable subset B with $V = \overline{\mathbb{K}\text{-span}(B)}$.
To show: V has a countable dense set $\mathbb{C}$.
Let $\mathbb{F} = \mathbb{Q}$ if $\mathbb{K} = \mathbb{R}$ and let $\mathbb{F} = \mathbb{Q} + i\mathbb{Q}$ if $\mathbb{K} = \mathbb{C}$.
Let $C = \mathbb{F}\text{-span}(B)$.
Then C is countable and $\overline{\mathbb{F}} = \mathbb{K}$ so that

$$\overline{C} = \overline{\mathbb{F}\text{-span}(B)} \supseteq \overline{\overline{\mathbb{F}}\text{-span}(B)} = \overline{\mathbb{K}\text{-span}(B)} = V.$$

So $\overline{C} = V$.
So C is dense in V. □