20 Tutorial 1: Proof machine

Work through proof of if W is complete the $B(V,W)$ is complete and put the reason why each line appears where it appears. The possible reasons are:

(a) (Proof type II) Assume the ifs
(b) (Proof type II) To show the thens
(c) (Rewriting) This is the definition of _______.
(d) (Proof type III) To show something exists, construct it.
(e) (Proof type III) To show the construction is valid.
(f) (Proof type I) Compute the left hand side.
(g) (Proof type I) Compute the right hand side.

Practice this proof so that you can do it without referring to notes.

20.1 If W is complete then $B(V,W)$ is complete

Theorem 20.1. *Let $(V, \| \|)$ and $(W, \| \|)$ be normed vector spaces and let*

$$B(V,W) = \{\textit{linear transformations } T\colon V \to W \mid \|T\| < \infty\} \qquad \textit{where}$$

$$\|T\| = \sup\left\{\frac{\|Tv\|}{\|v\|} \mid v \in V \textit{ and } v \neq 0\right\}.$$

If W is complete then $B(V,W)$ is complete.

Proof. To show: If W is complete then $B(V,W)$ is complete.
Assume W is complete.
To show: If $T_1, T_2, \ldots$ is a Cauchy sequence in $B(V,W)$ then $T_1, T_2, \ldots$ converges.
Assume $T_1\colon V \to W$, $T_2\colon V \to W$, ... is a Cauchy sequence in $B(V,W)$.
To show: There exists $T\colon V \to W$ with $T \in B(V,W)$ such that $\lim_{n\to\infty} T_n = T$.
Define $T\colon V \to W$ by

$$T(x) = \lim_{n\to\infty} T_n(x).$$

To show: (a) If $x \in V$ then $T(x)$ exists.
(b) $T \in B(V,W)$.
(c) $\lim_{n\to\infty} T_n = T$.

(a) Assume $x \in V$.
To show: $\lim_{n\to\infty} T_n(x)$ exists.
To show: $T_1(x), T_2(x), \ldots$ converges in W.
Since W is complete,
to show: $T_1(x), T_2(x), \ldots$ is Cauchy.
To show: If $\epsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $r, s \in \mathbb{Z}_{\geq N}$ then $\|T_r(x) - T_s(x)\| < \epsilon$.
Assume $\epsilon \in \mathbb{R}_{>0}$.
Using that $T_1, T_2, \ldots$ is Cauchy, let N be such that
if $r, s \in \mathbb{Z}_{\geq N}$ then $\|T_r - T_s\| < \frac{\epsilon}{\|x\|}$.
To show: If $r, s \in \mathbb{Z}_{\geq N}$ then $\|T_r(x) - T_s(x)\| < \epsilon$.

Assume $r, s \in \mathbb{Z}_{\ge N}$.
To show: $\|T_r(x) - T_s(x)\| < \epsilon$.

$$\|T_r(x) - T_s(x)\| \le \|T_r - T_s\| \cdot \|x\| < \frac{\epsilon}{\|x\|} \cdot \|x\| = \epsilon.$$

So $T_1(x), T_2(x), \ldots$ is Cauchy and, since W is complete, $T_1(x), T_2(x), \ldots$ converges.
So $T(x) = \lim_{n\to\infty} T_n(x)$ exists.

(b) To show: $T \in B(V, W)$.
To show: (ba) T is a linear transformation.
(bb) $\|T\| < \infty$.

(ba) To show: (baa) If $x_1, x_2 \in V$ then $T(x_1 + x_2) = T(x_1) + T(x_2)$.
(bab) If $c \in \mathbb{K}$ and $x \in V$ then $T(cx) = cT(x)$.

(baa) Assume $x_1, x_2 \in V$.
To show: $T(x_1 + x_2) = T(x_1) + T(x_2)$.
Since each T_n is a linear transformation and since

addition $\begin{matrix} +: & W \times W & \to & W \\ & (w_1, w_2) & \mapsto & w_1 + w_2 \end{matrix}$ is continuous in W, then

$$\begin{aligned} T(x_1 + x_2) &= \lim_{n\to\infty} T_n(x_1 + x_2) = \lim_{n\to\infty} (T_n(x_1) + T_n(x_2)) \\ &= \lim_{n\to\infty} T_n(x_1) + \lim_{n\to\infty} T_n(x_2) = T(x_1) + T(x_2). \end{aligned}$$

(bab) Assume $c \in \mathbb{K}$ and $x \in V$.
To show: $T(cx) = cT(x)$.
Since each T_n is a linear transformation and since

scalar mutliplication $\begin{matrix} \mathbb{K} \times W & \to & W \\ (c, w) & \mapsto & cw \end{matrix}$ is continuous in W,

$$T(cx) = \lim_{n\to\infty} T_n(cx) = \lim_{n\to\infty} cT_n(x) = c \lim_{n\to\infty} T_n(x) = cT(x).$$

So T is a linear transformation.

(bb) To show: $\|T\| < \infty$.
To show: $\|T\| = \sup\left\{ \frac{\|Tx\|}{\|x\|} \mid x \in V \right\}$ exists in $\mathbb{R}_{\ge 0}$.
Since $\| \ \| \colon W \to \mathbb{R}_{\ge 0}$ is continuous,

$$\begin{aligned} \|Tx\| &= \|\lim_{n\to\infty} T_n(x)\| = \lim_{n\to\infty} \|T_n(x)\| \\ &\le \lim_{n\to\infty} \|T_n\| \cdot \|x\| = \|x\| \big(\lim_{n\to\infty} \|T_n\| \big). \end{aligned}$$

By assumption, the sequence $T_1, T_2, \ldots$ is Cauchy and thus, since $\|T_r\| - \|T_s\| \le \|T_r - T_s\|$, the sequence $\|T_1\|, \|T_2\|, \ldots$ is Cauchy.
Since $\mathbb{R}_{\ge 0}$ is complete, $\lim_{n\to\infty} \|T_n\|$ exists.
So

$$\|T\| = \sup\left\{ \frac{\|Tx\|}{\|x\|} \mid x \in V \right\} \le \lim_{n\to\infty} \|T_n\|,$$

and the right hand side exists in $\mathbb{R}_{\ge 0}$.
So $\|T\| < \infty$.

So $T \in B(V, W)$.

(c) To show: $\lim_{n\to\infty} T_n = T$.

To show: If $\epsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq N}$ then $\|T - T_n\| < \epsilon$.
Assume $\epsilon \in \mathbb{R}_{>0}$.
To show: There exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq N}$ then $\|T - T_n\| < \epsilon$.
Using that the sequence $T_1, T_2, \ldots$ is Cauchy,
let $N \in \mathbb{Z}_{>0}$ be such that if $m, n \in \mathbb{Z}_{\geq N}$ then $\|T_m - T_n\| < \frac{\epsilon}{2}$.
To show: If $n \in \mathbb{Z}_{\geq N}$ then $\|T - T_n\| < \epsilon$.
Assume $n \in \mathbb{Z}_{\geq N}$.
To show: $\|T - T_n\| < \epsilon$.
To show: $\sup\left\{ \frac{\|(T-T_n)(x)\|}{\|x\|} \;\Big|\; x \in V \text{ and } x \neq 0 \right\} < \epsilon$.
Assume $x \in V$ and $x \neq 0$.
To show: $\frac{\|(T-T_n)(x)\|}{\|x\|} < \frac{\epsilon}{2}$.
To show: $\frac{\|T(x) - T_n(x)\|}{\|x\|} < \frac{\epsilon}{2}$.
To show: $\frac{\| \lim_{m\to\infty} T_m(x) - T_n(x)\|}{\|x\|} < \frac{\epsilon}{2}$.
Using that $\|\ \| \colon W \to \mathbb{R}_{\geq 0}$ is continuous, To show: $\frac{\| \lim_{m\to\infty} T_m(x) - T_n(x)\|}{\|x\|} < \frac{\epsilon}{2}$.
To show: There exists $M \in \mathbb{Z}_{>0}$ such that if $m \in \mathbb{Z}_{\geq M}$ then $\frac{\|T_m(x) - T_n(x)\|}{\|x\|} < \frac{\epsilon}{2}$.
Let $M = N$.
To show: If $m \in \mathbb{Z}_{\geq M}$ then $\frac{\|T_m(x) - T_n(x)\|}{\|x\|} < \frac{\epsilon}{2}$.
Assume $m \in \mathbb{Z}_{\geq M}$.
To show: $\frac{\|T_m(x) - T_n(x)\|}{\|x\|} < \frac{\epsilon}{2}$.
Since $m, n \in \mathbb{Z}_{\geq N}$ then

$$\frac{\epsilon}{2} > \|T_m - T_n\| = \sup\left\{ \frac{\|T_m(y) - T_n(y)\|}{\|y\|} \;\Big|\; y \in V \text{ and } y \neq 0 \right\} \geq \frac{\|T_m(x) - T_n(x)\|}{\|x\|}.$$

So $\frac{\|T_m(x) - T_n(x)\|}{\|x\|} < \frac{\epsilon}{2}$.
So $\sup\left\{ \frac{\|(T-T_n)(x)\|}{\|x\|} \;\Big|\; x \in V \text{ and } x \neq 0 \right\} \leq \frac{\epsilon}{2} < \epsilon$.
So $\|T - T_n\| < \epsilon$.
So $\lim_{n\to\infty} T_n = T$.
So $\|T - T_n\| \leq \frac{\epsilon}{2} < \epsilon$.
So $\lim_{n\to\infty} \|T - T_n\| = 0$.
So $\lim_{n\to\infty} T_n = T$.

□