33 Tutorial 4: Spectral theorem

Learn to prove the following theorems, accurately, efficiently, using quality proof machine, without having to refer to notes. Learn to prove the following theorems, accurately, efficiently, using quality proof machine, without having to refer to notes. The first step of this process is to work through each and put the reason why each line appears where it appears. The possible reasons are:

(a) (Proof type II) Assume the ifs
(b) (Proof type II) To show the thens
(c) (Rewriting) This is the definition of _______.
(d) (Proof type III) To show something exists, construct it.
(e) (Proof type III) To show the construction is valid.
(f) (Proof type I) Compute the left hand side.
(g) (Proof type I) Compute the right hand side.

Practice each proof so that you can do it efficiently without referring to notes.

33.0.1 Identifying eigenvectors

Proposition 33.1. *Let H be a Hilbert space and let $\lambda \in \mathbb{C}$.*

(a) Let $T\colon H \to H$ be a linear operator. Then

$$T \text{ has an eigenvector of eigenvalue } \lambda \quad \text{if and only if} \quad \lambda - T \text{ is not injective.}$$

(b) (Fredholm's theorem) Let $T\colon H \to H$ be a compact linear operator. Then

$$\lambda - T \text{ is injective} \qquad \text{if and only if} \qquad \lambda - T \text{ is bijective.}$$

33.0.2 Finding $\|T\|$

Theorem 33.2. *Let H be a Hilbert space and let $T\colon H \to H$ be a bounded self adjoint linear operator.*

$$\sup\{|\langle Tu, u\rangle| \mid \|u\| = 1\} = \|T\|.$$

If $(u_1, u_2, \ldots)$ is a sequence in $\{u \in H \mid \|u\| = 1\}$ such that

$$\lim_{n\to\infty} |\langle Tu_n, u_n\rangle| = \|T\| \qquad \text{then} \qquad \lim_{n\to\infty} (T-\lambda)u_n = 0.$$

33.0.3 $\|T\|$ is the largest eigenvalue

Theorem 33.3. *Let H be a Hilbert space and let $T\colon H \to H$ be a nonzero compact self adjoint linear operator. Let $(u_1, u_2, \ldots)$ be a sequence in $\{u \in H \mid \|u\| = 1\}$ such that*

$$\lim_{n\to\infty} |\langle Tu_n, u_n\rangle| = \|T\| \qquad \text{and let } y \text{ be a cluster point of } Tu_1, Tu_2, \ldots.$$

Then

$$\|y\| = \|T\|, \qquad \frac{|\langle Ty, y\rangle|}{\|y\|^2} = \|T\| \qquad \text{and} \qquad Ty = \|T\|\, y.$$

33.0.4 Eigenspaces of compact self adjoint operators

Theorem 33.4. *Let $T: H \to H$ be a self adjoint operator. For $\lambda \in \mathbb{K}$ let*

$$H_\lambda = \{v \in H \mid Tv = \lambda v\}.$$

(a) If $H_\lambda \neq 0$ then $\lambda \in \mathbb{R}$.

(b) If $\lambda \neq \gamma$ then $H_\lambda \perp H_\gamma$.

(c) If T is compact and $\lambda \neq 0$ then H_λ is finite dimensional.

(d) If T is compact then H is the closure of

$$\bigoplus_{\lambda \in \sigma_p(T)} H_\lambda \qquad \text{where} \quad \sigma_p(T) = \{\lambda \in \mathbb{K} \mid H_\lambda \neq 0\}.$$

33.0.5 Orthonormal bases of eigenvectors

Theorem 33.5. *Let H be a Hilbert space with a countable dense set and let $T: H \to H$ be a bounded self adjoint compact operator. Then there exists an orthonormal basis of eigenvectors of H.*

34 Tutorial 4: Solutions

34.1 Identifying eigenvectors

Proposition 34.1. *Let H be a Hilbert space and let $\lambda \in \mathbb{C}$.*

(a) Let $T: H \to H$ be a linear operator. Then

$$T \text{ has an eigenvector of eigenvalue } \lambda \quad \text{if and only if} \quad \lambda - T \text{ is not injective.}$$

(b) (Fredholm's theorem) Let $T: H \to H$ be a compact linear operator. Then

$$\lambda - T \text{ is injective} \qquad \text{if and only if} \qquad \lambda - T \text{ is bijective.}$$

Proof.

(a) $\Rightarrow$: Assume there exists an eigenvector v of T of eigenvalue λ.
To show: $\lambda - T$ is not injective.
To show: There exists $h \in H$ such that $(\lambda - T)(h) = 0$.
Let $h = v$.

$$(\lambda - T)(h) = (\lambda - T)(v) = \lambda v - \lambda v = 0.$$

So $\lambda - T$ is not injective.

(a) $\Leftarrow$: Assume that $T - \lambda$ is not injective.
Then there exists $x, y \in H$ with $x \neq y$ such that $(T - \lambda)(x) = (T - \lambda)(y)$.
So $0 = (T - \lambda)(x) - (T - \lambda)(y) = (T - \lambda)(x - y)$.
Since $x \neq y$ then $v = x - y \neq 0$ and $(T - \lambda)(v) = 0$.
So there exists $v \in H$ such that $v \neq 0$ and $T(v) = \lambda v$.
So T has an eigenvector of eigenvalue λ.

(b) $\Leftarrow$: Assume that $\lambda \in \mathbb{C}$ and $\lambda - T$ is bijective.
By the definition of bijective, $\lambda - T$ is injective (and surjective).

(b) $\Rightarrow$: Assume that $T: H \to H$ is a linear operator and $\lambda \in \mathbb{C}$ and $\lambda - T$ is injective.
To show: If T is compact then $\lambda - T$ is bijective.
To show: If T is compact then $\lambda - T$ is surjective.
To show: If $\mathrm{im}(\lambda - T) \neq H$ then T is not compact.
Assume $(\lambda - T)(H) \neq H$.
For $k \in \mathbb{Z}_{>0}$ let $W_k = (\lambda - T)^k(H)$ so that $W_{k+1} = (\lambda - T)^{k+1}(H) \subsetneq (\lambda - T)^k(H) = W_k$ and

$$H \supsetneq W_1 \subsetneq W_2 \supsetneq \cdots .$$

Let $e_k \in W_n \cap W_{n+1}^{\perp}$ with $\|e_k\| = 1$. Then

$$\|Te_m - Te_n\| \geq \|\lambda e_m\| = |\lambda|.$$

So $(Te_1, Te_2, \cdots)$ is not Cauchy.
So $(Te_1, Te_2, \cdots)$ does not have a convergent subsequence.
So T is not compact. CLEAN UP CLEAN UP. $\square$

34.2 Finding $\|T\|$

Theorem 34.2. *Let H be a Hilbert space and let $T\colon H \to H$ be a bounded self adjoint linear operator. Let*

$$\lambda = \sup\{\ |\langle Tu, u\rangle| \mid \|u\| = 1\}$$

Let $(u_1, u_2, \ldots)$ be a sequence in $\{u \in H \mid \|u\| = 1\}$ such that

$$(|\langle Tu_1, u_1\rangle|, |\langle Tu_2, u_2\rangle|, \ldots) \quad \textit{is increasing} \quad \textit{and} \quad \lim_{n\to\infty} |\langle Tu_n, u_n\rangle| = \lambda.$$

Then

$$\lambda = \|T\| \qquad \textit{and} \qquad \lim_{n\to\infty} (T-\lambda)u_n = 0.$$

Proof. (a) Let $\lambda = \sup\{|\langle Tu, u\rangle| \mid \|u\| = 1\}$.
To show: $\|T\| = \lambda$.
To show: (aa) $\|T\| \geq \lambda$.
(ab) $\|T\| \leq \lambda$.

(aa) Assume $u \in H$ and $\|u\| = 1$.

By Cauchy-Schwarz,

$$|\langle Tu, u\rangle| \leq \|Tu\| \cdot \|u\| \leq \|T\| \cdot \|u\| \cdot \|u\| = \|T\|.$$

So $\lambda \leq \|T\|$.

(ab) Let $x \in H$ with $\|x\| = 1$. Let

$$y = \frac{Tx}{\|Tx\|} \qquad \text{so that} \qquad \|y\| = 1.$$

Since T is self adjoint then $\langle Tx, y\rangle \in \mathbb{R}$ and $\langle y, Tx\rangle = \langle Tx, y\rangle$ and

$$\begin{aligned}
&\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle \\
&\quad = \langle Tx, x\rangle + \langle Tx, y\rangle + \langle Ty, x\rangle + \langle Ty, y\rangle - \langle Tx, x\rangle + \langle Tx, y\rangle + \langle Ty, x\rangle - \langle Ty, y\rangle \\
&\quad = 4\langle Tx, y\rangle = 4\frac{\langle Tx, Tx\rangle}{\|Tx\|} = 4\|Tx\|.
\end{aligned}$$

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Then

$$\begin{aligned}
4\|Tx\| &= \big|\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle\big| \\
&\leq \big|\langle T(x+y), x+y\rangle\big| + \big|\langle T(x-y), x-y\rangle\ \big| \\
&= \Big|\langle \frac{T(x+y)}{\|x+y\|}, \frac{x+y}{\|x+y\|}\rangle\Big| \|x+y\|^2 + \Big|\langle \frac{T(x-y)}{\|x-y\|}, \frac{x-y}{\|x-y\|}\rangle\ \Big| \|x-y\|^2 \\
&\leq \lambda\|x+y\|^2 + \lambda\|x-y\|^2 \\
&= \lambda(2\|x\|^2 + 2\|y\|^2) = 4\lambda.
\end{aligned}$$

So $\|T\| \leq \lambda$.

So $\|T\| = \lambda$.
(b) Let $(u_1, u_2, \ldots)$ be a sequence in $\{u \in H \mid \|u\| = 1\}$ such that

$$(|\langle Tu_1, u_1\rangle|, |\langle Tu_2, u_2\rangle|, \ldots) \quad \text{is increasing} \quad \text{and} \quad \lim_{n\to\infty} |\langle Tu_n, u_n\rangle| = \lambda.$$

To show: $\lim_{k\to\infty} (T - \lambda)u_k = 0$.
To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $k \in \mathbb{Z}_{\geq N}$ then $\|Tu_k - \lambda u_k\|^2 \leq \varepsilon^2$.
Assume $\varepsilon \in \mathbb{R}_{>0}$.
To show: There exists $N \in \mathbb{Z}_{>0}$ such that if $k \in \mathbb{Z}_{\geq N}$ then $\|Tu_k - \lambda u_k\|^2 \leq \varepsilon^2$.
Let $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq N}$ then $|\langle Tu_n, u_n\rangle| > \lambda - \dfrac{\varepsilon^2}{2\lambda}$.
To show: If $k \in \mathbb{Z}_{\geq N}$ then $\|Tu_k - \lambda u_k\|^2 \leq \varepsilon^2$.
Assume $k \in \mathbb{Z}_{\geq N}$.
To show: $\|Tu_k - \lambda u_k\|^2 \leq \varepsilon^2$.

$$\begin{aligned}
\|Tu_k - \lambda u_k\|^2 &= \langle Tu_k - \lambda u_k, Tu_k - \lambda u_k\rangle \\
&= \langle Tu_k, Tu_k\rangle - 2\langle Tu_k, u_k\rangle\lambda + \lambda^2\langle u_k, u_k\rangle \\
&= \|Tu_k\|^2 - 2\lambda\langle Tu_k, u_k\rangle + \lambda^2 \\
&\leq \|T\|^2 - 2\lambda\langle Tu_k, u_k\rangle + \lambda^2 \\
&= \lambda^2 - 2\lambda\langle Tu_k, u_k\rangle + \lambda^2 \\
&< 2\lambda^2 - 2\lambda\left(\lambda - \frac{\varepsilon^2}{2\lambda}\right) = \varepsilon^2.
\end{aligned}$$

So $\lim_{k\to\infty} (T - \lambda)u_k = 0$. □

34.3 $\|T\|$ is the largest eigenvalue

Theorem 34.3. *Let H be a Hilbert space and let $T\colon H \to H$ be a nonzero bounded compact self adjoint linear operator. Let*

$$\lambda = \sup\{|\langle Tu, u\rangle| \mid \|u\| = 1\}.$$

Let $(u_1, u_2, \ldots)$ be a sequence in $\{u \in H \mid \|u\| = 1\}$ such that

$$\lim_{n\to\infty} |\langle Tu_n, u_n\rangle| = \lambda.$$

Let y be a cluster point of the sequence $(Tu_1, Tu_2, \ldots)$ and let $(u_{n_1}, u_{n_2}, \ldots)$ be a subsequence of $(u_1, u_2, \ldots)$ such that

$$\lim_{k\to\infty} Tu_{n_k} = y.$$

Then

(a) $\lambda = \|T\|$.

(b) $\lim_{k\to\infty} (T - \lambda)u_k = 0$.

(c) $\lim_{k\to\infty} u_{n_k} = \dfrac{y}{\lambda}$.

(d) $\|y\| = \|T\|$.

(e) $Ty = \lambda y$.

(f) $\langle Ty, y\rangle = \langle \lambda y, y\rangle = \lambda \|y\|^2$.

Proof. The proof of (a) and (b) are as in the previous Proposition.
(c) To show: $\lim\limits_{k\to\infty} u_{n_k} = \dfrac{y}{\lambda}$.
To show: $\lim\limits_{k\to\infty} \lambda u_{n_k} = y$.
To show: If $\epsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $k \in \mathbb{Z}_{\geq N}$ then $\|y - \lambda u_{n_k}\| < \epsilon$.
Assume $\epsilon \in \mathbb{R}_{>0}$.
Let $N_1 \in \mathbb{Z}_{>0}$ such that if $k \in \mathbb{Z}_{\geq N_1}$ then $\|y - Tu_{n_k}\| < \frac{\epsilon}{10}$.
Using part (b), which says that $\lim_{k\to\infty}(T-\lambda)u_{n_k} = 0$,
let $N_2 \in \mathbb{Z}_{>0}$ such that if $k \in \mathbb{Z}_{\geq N_2}$ then $\|Tu_{n_k} - \lambda u_{n_k}\| < \frac{\epsilon}{10}$.
Let $N = \max\{N_1, N_2\}$.
Then

$$\|y - \lambda u_{n_k}\| \leq \|y - Tu_{n_k}\| + \|Tu_{n_k} - \lambda u_{n_k}\| < \frac{\epsilon}{10} + \frac{\epsilon}{10} < \epsilon.$$

So $\lim\limits_{k\to\infty} \lambda u_{n_k} = y$.
(d) Since $\|\cdot\|\colon H \to \mathbb{R}_{\geq 0}$ is continuous then

$$\|y\| = \|\lim_{k\to\infty} \lambda u_{n_k}\| = \lim_{k\to\infty} \|\lambda u_{n_k}\| = \lim_{k\to\infty} |\lambda| \cdot \|u_{n_k}\| = |\lambda| = \|T\|.$$

(e) Since T is bounded then T is continuous. Using that T is continuous and that $\|\cdot\|\colon H \to \mathbb{R}_{\geq 0}$ is continuous then

$$\|Ty - \lambda y\| = \|T \lim_{k\to\infty} \lambda u_{n_k} - \lambda y\| = \lim_{k\to\infty} \lambda \|Tu_{n_k} - y\| = \lambda \lim_{k\to\infty} \|Tu_{n_k} - y\| = 0.$$

Since $\|y\| = |\lambda| = \|T\|$ and T is not the 0 operator then $y \neq 0$.
Since $|Ty - \lambda y\| = 0$ then $Ty - \lambda y = 0$.
So $Ty = \lambda y$.
(f) Using that $Ty = \lambda y$ then

$$\langle Ty, y\rangle = \langle \lambda y, y\rangle = \lambda \|y\|^2.$$

□

34.4 Eigenspaces of compact self adjoint operators

Theorem 34.4. *Let $T\colon H \to H$ be a self adjoint operator. For $\lambda \in \mathbb{K}$ let*

$$H_\lambda = \{v \in H \mid Tv = \lambda v\} \qquad \text{and let} \quad \sigma_p(T) = \{\lambda \in \mathbb{K} \mid H_\lambda \neq 0\}.$$

(a) If $H_\lambda \neq 0$ then $\lambda \in \mathbb{R}$.

(b) If $\lambda \neq \gamma$ then $H_\lambda \perp H_\gamma$.

(c) If T is compact and $\lambda \neq 0$ then H_λ is finite dimensional.

(d) If $\lambda_1, \lambda_2, \ldots$ is a sequence of distinct eigenvalues in $\sigma_p(T)$ then $\lim\limits_{n\to\infty} \lambda_n = 0$.

(e) If T is compact then H is the closure of

$$\bigoplus_{\lambda \in \sigma_p(T)} H_\lambda$$

Proof. (a) Let $v \in H_\lambda$ with $v \neq 0$.
Since T is self adjoint then $\lambda\langle v, v\rangle = \langle \lambda v, v\rangle = \langle Tv, v\rangle = \langle v, Tv\rangle = \overline{\lambda}\langle v, v\rangle$.
So $\lambda \in \mathbb{R}$.

(b) Assume $\lambda \neq \gamma$.
To show: $H_\lambda \perp H_\gamma$.
To show: If $x \in H_\lambda$ and $y \in H_\gamma$ then $\langle x, y\rangle = 0$.
Assume $x \in H_\lambda$ and $y \in H_\gamma$.
Since T is self adjoint, $\lambda\langle x, y\rangle = \langle Tx, y\rangle = \langle x, Ty\rangle = \gamma\langle x, y\rangle$.
So $(\lambda - \gamma)\langle x, y\rangle = 0$.
Since $\lambda - \gamma \neq 0$ then $\langle x, y\rangle = 0$.
So $H_\lambda \perp H_\gamma$.

(c) To show: If $\lambda \neq 0$ and H_λ is infinite dimensional then $T\colon H \to H$ is not compact.
Assume $\lambda \neq 0$ and H_λ is infinite dimensional.
Let $(e_1, e_2, \ldots)$ be an orthonormal sequence in H_λ.
If $m \neq n$ then

$$\|Te_m - Te_n\| = \|\lambda e_m - \lambda e_n\| = |\lambda| \cdot \|e_m - e_n\| = |\lambda| \cdot \sqrt{2}.$$

So no subsequence of $(Te_1, Te_2, \ldots)$ is Cauchy.
So $(Te_1, Te_2, \ldots)$ does not have a convergent subsequence.
So T is not compact.

(d) Let $(\lambda_1, \lambda_2, \ldots)$ be a sequence of distinct eigenvalues in $\sigma_p(T)$.
To show: If T is compact then $\lim_{n\to\infty} \lambda_n = 0$.
To show: If $\lim_{n\to\infty} \lambda_n \neq 0$ then T is not compact.
Assume $\lim_{n\to\infty} \lambda_n \neq 0$.
To show: T is not compact.
Let $u_n \in H_{\lambda_n}$ with $\|u_n\| = 1$ and let $W_n = \mathrm{span}\{u_1, \ldots, u_n\}$ so that

$$W_1 \subseteq W_2 \subseteq \cdots .$$

For $n \in \mathbb{Z}_{>0}$ let $e_n \in W_n \cap W_{n-1}^\perp$. DOESN'T u_n ALREADY SATIFY THIS SINCE EIGENSPACES ARE ORTHOGONAL?
Then

$$\|Te_n - Te_m\| = \|(T - \lambda_n)e_n - (T - \lambda_m)e_m + (\lambda_n e_n - \lambda_m e_m)\| \geq \|\lambda_n e_n\| = |\lambda_n|.$$

Since $\lim_{n\to\infty} \lambda_n \neq 0$ then the sequence $(Te_1, Te_2, \ldots)$ is not Cauchy.
So the sequence $(Te_1, Te_2, \ldots)$ does not have a convergent subsequence.
So T is not compact.

(e) PUT IN THE PROOF OF THIS. □

34.5 Orthonormal bases of eigenvectors

Theorem 34.5. *Let H be a Hilbert space with a countable dense set and let $T\colon H \to H$ be a bounded self adjoint compact operator. Then there exists an orthonormal basis of eigenvectors of H.*

Proof. For $\lambda \in \sigma_p(T)$ let B_λ be an orthonormal basis of H_λ, constructed by Gram-Schmidt.
Let

$$B = \bigcup_{\lambda\in\sigma_p(T)} B_\lambda. \qquad \text{Since} \qquad H = \overline{\Big(\bigoplus_{\lambda\in\sigma_p(T)} H_\lambda\Big)}$$

and each element of B_λ is an eigenvector of T of eigenvalue λ then B is an orthonormal basis of eigenvectors of T. $\square$