12 Sets, functions and relations

12.1 Sets and functions

12.1.1 Sets

A *set* is a collection of objects which are called *elements*.
Write

$$s \in S \text{ if } s \text{ is an element of the set } S.$$

- The *empty set* $\emptyset$ is the set with no elements.
- A *subset* T of a set S is a set T such that if $t \in T$ then $t \in S$.

Write

$$\begin{aligned} &T \subseteq S \text{ if } T \text{ is a subset of } S, \text{ and} \\ &T = S \text{ if the set } T \text{ is equal to the set } S. \end{aligned}$$

Let S and T be sets.

- The *union of S and T* is the set $S \cup T$ of all u such that $u \in S$ or $u \in T$,
$$S \cup T = \{u \mid u \in S \text{ or } u \in T\}.$$
- The *intersection of S and T* is the set $S \cup T$ of all u such that $u \in S$ and $u \in T$,
$$S \cap T = \{u \mid u \in S \text{ and } u \in T\}.$$
- The *product S and T* is the set $S \times T$ of all ordered pairs (s,t) where $s \in S$ and $t \in T$,
$$S \times T = \{(s,t) \mid s \in S \text{ and } t \in T\}.$$

The sets S and T are *disjoint* if $S \cap T = \emptyset$.
The set S is a *proper subset* of T if $S \subseteq T$ and $S \neq T$.

12.1.2 Functions

Functions are for comparing sets.

Let S and T be sets. A *function from S to T* is a subset $\Gamma_f \subseteq S \times T$ such that

$$\text{if } s \in S \quad \text{then} \quad \text{there exists a unique } t \in T \text{ such that } (s,t) \in \Gamma_f.$$

Write

$$\Gamma_f = \{(s, f(s)) \mid s \in S\}$$

so that the function Γ_f can be expressed as

$$\text{an "assignment"} \qquad \begin{array}{rccc} f\colon & S & \to & T \\ & s & \mapsto & f(s) \end{array}$$

which must satisfy

(a) If $s \in S$ then $f(s) \in T$, and

(b) If $s_1, s_2 \in S$ and $s_1 = s_2$ then $f(s_1) = f(s_2)$.

Let S and T be sets.

- Two functions $f\colon S \to T$ and $g\colon S \to T$ are *equal* if they satisfy
$$\text{if } s \in S \quad \text{then} \quad f(s) = g(s).$$
- A function $f\colon S \to T$ is *injective* if f satisfies the condition
$$\text{if } s_1, s_2 \in S \text{ and } f(s_1) = f(s_2) \quad \text{then} \quad s_1 = s_2.$$
- A function $f\colon S \to T$ is *surjective* if f satisfies the condition
$$\text{if } t \in T \quad \text{then} \quad \text{there exists } s \in S \text{ such that } f(s) = t.$$
- A function $f\colon S \to T$ is *bijective* if f is both injective and surjective.

12.1.3 Composition of functions

Let $f\colon S \to T$ and $g\colon T \to U$ be functions. The *composition* of f and g is the function

$$g \circ f \quad \text{given by} \quad \begin{array}{rccc} g \circ f\colon & S & \to & U \\ & s & \mapsto & g(f(s)) \end{array}$$

Let S be a set. The *identity map on* S is the function given by

$$\begin{array}{rccc} \mathrm{id}_S\colon & S & \to & S \\ & s & \mapsto & s \end{array}$$

Let $f\colon S \to T$ be a function. The *inverse function to* f is a function

$$f^{-1}\colon T \to S \qquad \text{such that} \qquad f \circ f^{-1} = \mathrm{id}_T \quad \text{and} \quad f^{-1} \circ f = \mathrm{id}_S.$$

Theorem 12.1. *Let $f\colon S \to T$ be a function. An inverse function to f exists if and only if f is bijective.*

Let S and T be sets. The sets *S and T are isomorphic*, or *have the same cardinality*

$$\text{if there is a bijective function} \quad \varphi\colon S \to T.$$

Write $\mathrm{Card}(S) = \mathrm{Card}(T)$ if S and T have the same cardinality.

Notation: Let S be a set. Write

$$\mathrm{Card}(S) = \begin{cases} 0, & \text{if } S = \emptyset, \\ n, & \text{if } \mathrm{Card}(S) = \mathrm{Card}(\{1, 2, \ldots, n\}), \\ \infty, & \text{otherwise.} \end{cases}$$

Note that even in the cases where $\mathrm{Card}(S) = \infty$ and $\mathrm{Card}(T) = \infty$ it may be that $\mathrm{Card}(S) \neq \mathrm{Card}(T)$.

Let S be a set.

- The set S is *finite* if $\mathrm{Card}(S) \neq \infty$.
- The set S is *infinite* if $\mathrm{Card}(S)$ is not finite.

- The set S is *countable* if $\mathrm{Card}(S) = \mathrm{Card}(\mathbb{Z}_{>0})$.
- The set S is *countably infinite* if S is countable and infinite.
- The set S is *uncountable* if S is not countable.

Let $\mathcal{S}et$ be the set of sets. Define a relation $\sim$ on $\mathcal{S}et$ by

$$X \sim Y \qquad \text{if there exists a bijection } f\colon X \to Y.$$

The relation $\sim$ is an equivalence relation and $\mathrm{Card}(X)$ is the equivalence class of X. The set of *ordinals* is the set of equivalence classes of $\sim$,

$$\mathcal{O}rd = \{\mathrm{Card}(X) \mid X \in \mathcal{S}et\}.$$

Theorem 12.2. *Define a relation $\preceq$ on $\mathcal{S}et$ by*

$$X \preceq Y \qquad \textit{if there exists an injection } f\colon X \to Y.$$

(a) The relation $\preceq$ on $\mathcal{S}et$ gives a well defined relation $\le$ on $\mathcal{O}rd$,

$$\mathrm{Card}(X) \le \mathrm{Card}(Y) \quad \textit{if there exists an injection } f\colon X \to Y,$$

where $X \in \mathrm{Card}(X)$ and $Y \in \mathrm{Card}(Y)$.

(b) The relation $\le$ is a partial order on $\mathcal{O}rd$.

12.2 Relations, equivalence relations and partitions

Let S be a set.

- A *relation $\sim$ on S* is a subset $R_\sim$ of $S \times S$. Write $s_1 \sim s_2$ if the pair (s_1, s_2) is in the subset $R_\sim$ so that
$$R_\sim = \{(s_1, s_2) \in S \times S \mid s_1 \sim s_2\}.$$
- An *equivalence relation* on S is a relation $\sim$ on S such that
  (a) if $s \in S$ then $s \sim s$,
  (b) if $s_1, s_2 \in S$ and $s_1 \sim s_2$ then $s_2 \sim s_1$,
  (c) if $s_1, s_2, s_3 \in S$ and $s_1 \sim s_2$ and $s_2 \sim s_3$ then $s_1 \sim s_3$.

Let $\sim$ be an equivalence relation on a set S and let $s \in S$. The *equivalence class of s* is the set

$$[s] = \{t \in S \mid t \sim s\}.$$

A *partition of a set S* is a collection $\mathcal{P}$ of subsets of S such that

(a) If $s \in S$ then there exists $P \in \mathcal{P}$ such that $s \in P$, and

(b) If $P_1, P_2 \in \mathcal{P}$ and $P_1 \cap P_2 \neq \emptyset$ then $P_1 = P_2$.

Theorem 12.3.

(a) If S is a set and let $\sim$ be an equivalence relation on S then

$$\textit{the set of equivalence classes of} \quad \sim \quad \textit{is a partition of } S.$$

(b) If S is a set and $\mathcal{P}$ is a partition of S then

$$\textit{the relation defined by} \qquad s \sim t \quad \textit{if } s \textit{ and } t \textit{ are in the same } P \in \mathcal{P}$$

is an equivalence relation on S.

12.3 Some proofs

12.3.1 An inverse function to f exists if and only if f is bijective

Theorem 12.4. *Let $f\colon S \to T$ be a function. The inverse function to f exists if and only if f is bijective.*

Proof.

$\Rightarrow$: Assume $f\colon S \to T$ has an inverse function $f^{-1}\colon T \to S$.
To show: (a) f is injective.
(b) f is surjective.

(a) Assume $s_1, s_2 \in S$ and $f(s_1) = f(s_2)$.
To show: $s_1 = s_2$.

$$s_1 = f^{-1}f(s_1)) = f^{-1}f(s_2)) = s_2.$$

So f is injective.

(b) Let $t \in T$.
To show: There exists $s \in S$ such that $f(s) = t$.
Let $s = f^{-1}(t)$.
Then

$$f(s) = f(f^{-1}(t)) = t.$$

So f is surjective.

So f is bijective.

$\Leftarrow$: Assume $f\colon S \to T$ is bijective.
To show: f has an inverse function.
We need to define a function $\varphi\colon T \to S$.
Let $t \in T$.
Since f is surjective there eists $s \in S$ such that $f(s) = t$.
Define $\varphi(t) = s$.
To show: (a) φ is well defined.
(b) φ is an inverse function to f.

(a) To show: (aa) If $t \in T$ then $\varphi(t) \in S$.
(ab) If $t_1, t_2 \in T$ and $t_1 = t_2$ then $\varphi(t_1) = \varphi(t_2)$.

(aa) This follows from the definition of φ.
(ab) Assume $t_1, t_2 \in T$ and $t_1 = t_2$.
Let $s_1, s_2 \in S$ such that $f(s_1) = t_1$ and $f(s_2) = t_2$.
Since $t_1 = t_2$ then $f(s_1) = f(s_2)$.
Since f is injective this implies that $s_1 = s_2$.
So $\varphi(t_1) = s_1 = s_2 = \varphi(t_2)$.

So φ is well defined.

(b) To show: (ba) If $s \in S$ then $\varphi(f(s)) = s$.
(bb) If $t \in T$ then $f(\varphi(t)) = t$.

(ba) This follows from the definition of φ.

(bb) Assume $t \in T$.
Let $s \in S$ be such that $f(s) = t$.
Then
$$f(\varphi(t)) = f(s) = t.$$

So $\varphi \circ f$ and $f \circ \varphi$ are the identity functions on S and T, respectively.

So φ is an inverse function to f.

□

12.3.2 An equivalence relation on S and a partition of S are the same data

Theorem 12.5.

(a) If S is a set and let $\sim$ be an equivalence relation on S then

$$\textit{the set of equivalence classes of} \quad \sim \quad \textit{is a partition of } S.$$

(b) If S is a set and $\mathcal{P}$ is a partition of S then

$$\textit{the relation defined by} \qquad s \sim t \quad \textit{if } s \textit{ and } t \textit{ are in the same } P \in \mathcal{P}$$

is an equivalence relation on S.

Proof.

(a) To show: (aa) If $s \in S$ then s is in some equivalence class.
(ab) If $[s] \cap [t] \neq \emptyset$ then $[s] = [t]$.

(aa) Let $s \in S$.
Since $s \sim s$ then $s \in [s]$.

(ab) Assume $[s] \cap [t] \neq \emptyset$.
To show: $[s] = [t]$.
Since $[s] \cap [t] \neq \emptyset$ then there is an $r \in [s] \cap [t]$.
So $s \sim r$ and $r \sim t$.
By transitivity, $s \sim t$.
To show: (aba) $[s] \subseteq [t]$.
(abb) $[t] \subseteq [s]$.

(aba) Assume $u \in [s]$.
Then $u \sim s$.
We know $s \sim t$.
So, by transitivity, $u \sim t$.
Therefore $u \in [t]$.

So $[s] \subseteq [t]$.

(aba) Assume $v \in [t]$.
Then $v \sim t$.
We know $t \sim s$.
So, by transitivity, $v \sim s$.
Therefore $v \in [s]$.

So $[t] \subseteq [s]$.
So $[s] = [t]$.

So the equivalence classes partition S.

(b) To show: $\sim$ is an equivalence relation, i.e. that $\sim$ is reflexive, symmetric and transitive.

To show: (ba) If $s \in S$ then $s \sim s$.
(bb) If $s \sim t$ then $t \sim s$.
(bc) If $s \sim t$ and $t \sim u$ then $s \sim u$.

(ba) Since s and s are in the same S_α then $s \sim s$.

(bb) Assume $s \sim t$.

Then s and t are in the same S_α.

So $t \sim s$.

(bb) Assume $s \sim t$ and $t \sim u$.

Then s and t are in the same S_α and t and u are in the same S_α.

So $s \sim u$.

So $\sim$ is an equivalence relation.

□

12.4 Notes and references

Almost everything in mathematics is built from sets and functions. Groups, rings, fields, vector spaces ... are all sets endowed with additional functions which have special properties. In the society of mathematics, sets and functions are the individuals and the fascination is the way that the individuals, each one different from the others, interact.

Functions are the morphisms in the category $\mathcal{S}$et of sets and products are products in the category $\mathcal{S}$et of all sets. The set of sets $\mathcal{S}$et may or may not make sense to you: there are good reasons – study Russell's paradox, the Zermelo-Frenkel axioms and small categories to learn more.