MAST30026 Metric and Hilbert Spaces

Sample exam 2

Question 1. Let X be a topological space, and let A be a subset of X.

(a) Define the *closure* $\overline{A}$ of A. (Give a definition in terms of closed sets.)

(b) Show that $x \in \overline{A}$ if and only if every open neighbourhood of x intersects A.

(c) Using (b) or otherwise, show that if $f : X \to Y$ is a continuous map between topological spaces and $A \subset X$ then $f(\overline{A}) \subset \overline{f(A)}$.

Question 2.

(a) Let (X, d) and (Y, d') be metric spaces. Show that $d^*((x, y), (u, v)) = d(x, u) + d'(y, v)$ defines a metric on $X \times Y$.

(b) Prove that the map $f : (X, d) \to (X \times Y, d^*)$ given by $x \to (x, y_0)$ is an isometry from X to $f(X)$.

(c) Use (b) to deduce that if (X, d) and (Y, d') are connected spaces, then $(X \times Y, d^*)$ is a connected space. (Hint: If $X \times Y = U \cup V$ with U, V disjoint, open, show that $f(X) \subset U$ or $f(X) \subset V$. Repeat for different points y_0.)

Question 3.

(a) Let (X, d) be a metric space and let $\{f_n\}$ be a sequence of continuous functions, $f_n : X \to \mathbb{R}$, for $n \in \mathbb{Z}_{>0}$. Give the definition of uniform convergence of the sequence f_n to a function $f : X \to \mathbb{R}$.

(b) Prove that if $\{f_n\}$ converges uniformly to $f : X \to \mathbb{R}$, then f is a continuous function.

(c) Let $f_n(x) = \dfrac{1 - x^n}{1 + x^n}$ for $x \in [0, 1]$ and $n \in \mathbb{Z}_{>0}$. Find the pointwise limit f of the sequence $\{f_n\}$. Determine whether the sequence f_n is uniformly convergent to f or not on the interval $[0, 1]$. Give brief reasons for your answer.

(d) Is the sequence (f_n) uniformly convergent on the interval $[0, 1]$?

Question 4. Let $(l^2, < \cdot >)$ denote the Hilbert space of sequences $(a_1, a_2, \dots)$, satisfying $\sum_{n=1}^{\infty} |a_n|^2$ is convergent. The inner product is defined by

$$< (a_1, a_2, \dots), (b_1, b_2, \dots) >= \sum_{n=1}^{\infty} a_n \overline{b_n}$$

Let $T : l^2 \to l^2$ be a linear transformation.

(a) Define what it means for a set to be a Schauder basis for a separable Banach or Hilbert space.

You may assume that l^2 has a Schauder basis $\mathcal{S} = \{e_1, e_2 \dots\}$ where $e_1 = (1, 0, 0 \dots), e_2 = (0, 1, 0, \dots), \dots$.

(b) Show that T is a bounded linear operator if and only if the sequence $\|T(e_1)\|, \|T(e_2)\|, \ldots$ is bounded.

(c) If $Te_j = \sum_{n=1}^{\infty} c_{jn} e_n$, give a condition on the coefficients c_{jn} which is necessary and sufficient for T to be self adjoint. Give reasons for your answer.

Question 5.

(a) Define compactness for a metric space (X, d).

1. Let ℓ^∞ be the set of bounded real sequences with the supremum metric.

(b) Consider the following metric spaces. Which of these spaces are compact? Give brief explanations.

(1) The circle $\{(x, y) : x^2 + y^2 = 1\}$ with the metric induced from $\mathbb{R}^2$;
(2) The open disk $\{(x, y) : x^2 + y^2 < 1\}$ with the metric induced from $\mathbb{R}^2$.
(3) The closed unit ball in the space ℓ^∞.

Question 6. Suppose that $(H, < \cdot >)$ is a real Hilbert space.

(a) Prove that the functional $f : H \to \mathbb{R}$ given by $f(x) =< x, v >$ is a bounded linear operator, where v is a fixed element of H. Compute $\|f\|$ for this functional.

(b) State the Riesz representation theorem.

(c) Suppose that $T : V \to W$ is a bounded linear operator between Banach spaces V, W. Use the Riesz representation theorem to give the construction of an adjoint operator to T. Prove that the adjoint operator is uniquely defined by your construction and is a linear operator. (You don't have to prove that the adjoint operator is bounded).

Question 7.

(a) Give the definition of a compact self adjoint linear operator $T : V \to W$ where V, W are Hilbert spaces.

(b) State the spectral expansion theorem for compact self adjoint linear operators.

(c) Prove that the sum of two compact self adjoint linear operators is compact and self adjoint. (Hint: You may use the fact that if A, B are compact subsets of a normed space, then $A + B = \{a + b : a \in A, b \in B\}$ is compact.)

Question 8.

(a) State the Banach fixed point theorem.

A mapping $f : \mathbb{R} \to \mathbb{R}$ is defined as a *contraction* if there exists a constant c with $0 \le c < 1$ such that $|f(x) - f(y)| \le c|x - y|$ for all $x, y \in \mathbb{R}$.

(b) (1) Use (a) to show that the equation $x + f(x) = a$ has a unique solution for each $a \in \mathbb{R}$.
(2) Deduce that $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = x + f(x)$ is a bijection. (This should be easy).
(3) Show that F is continuous.
(4) Show that F^{-1} is continuous. (Hence F is a homeomorphism.)