MAST30026 Metric and Hilbert Spaces

Sample exam 3

Question 1. (10 Marks) Let (X, d) be a metric space.

(a) State the definition of sequential compactness.

(b) Suppose that X is sequentially compact and nonempty. Given $\epsilon > 0$ prove that there exists a finite set $x_1, \ldots, x_n \in X$ such that $\{B_\epsilon(x_i)\}_{i=1}^n$ covers X.

You must prove (b) directly from the definition of sequential compactness.

Question 2. (20 marks) Let X be a topological space, $\Delta = \{(x, x) \in X \times X \mid x \in X\}$. Prove

(a) X is Hausdorff if and only if Δ is closed in $X \times X$.

(b) If X is Hausdorff and $f: Y \to X$ and $g: Y \to X$ are continuous maps and $A \subseteq Y$ is dense then $f = g$ if and only if $f(a) = g(a)$ for all $a \in A$.

Question 3. (20 marks) Let X be a locally compact Hausdorff space and Y, Z topological spaces. Let

$$\pi_Y : Y \times Z \to Y, \qquad \pi_Z : Y \times Z \to Z$$

be the projection maps. Prove that the function

$$\begin{array}{ccc} \mathrm{Cts}(X, Y \times Z) & \longrightarrow & \mathrm{Cts}(X, Y) \times \mathrm{Cts}(X, Z) \\ f & \longmapsto & (\pi_Y \circ f, \pi_Z \circ f) \end{array}$$

is a homeomorphism, with respect to the compact-open topology. You may assume the universal property of the product, and the adjunction property for the compact-open topology (including continuity of evaluation maps).

Question 4. (20 marks) Let $(V, \| \|)$ be a normed space over a field of scalars $\mathbb{F}$ (which recall denotes either $\mathbb{R}$ or $\mathbb{C}$).

(a) Prove that $\| \|: V \to \mathbb{F}$ is uniformly continuous.

Prove that V is a topological vector space by proving

(b) The addition $V \times V \to V$ is continous.

(c) The scalar multiplication $\mathbb{F} \times V \to V$ is continuous.

You may prove continuity using either the product topology or the product metric.

Question 5. (20 marks) Let $(V, \| \|)$ be a normed space over a field of scalars $\mathbb{F}$ and let $V^\vee$ denote the space of continuous linear maps $V \to \mathbb{F}$ with the operator norm. You may assume that this is a normed space. Prove that this space is *complete*, as follows:

(a) Given a Cauchy sequence $(T_n)_{n=0}^\infty$ in $V^\vee$ with respect to the operator norm, construct a candidate limit T as a function $T: V \to \mathbb{F}$.

(b) Prove that your candidate T is linear.

(b) Prove that your candidate T is bounded.

(d) Prove that $T_n \to T$ in the operator norm as $n \to \infty$.

Question 6. (20 marks) Let $(H, \langle\ \rangle)$ be a Hilbert space over $\mathbb{C}$.

(a) State the Cauchy-Schwartz inequality.

(b) Prove that for any $h \in H$ the function $\langle \cdot, h\rangle \colon H \to \mathbb{C}$ is continuous.

(c) Prove that if $\{u_i\}_{i\in I}$ is a set of vectors in H which span a vector subspace $U \subseteq H$ with the property that U is dense in H, then $H = 0$ if and only if $\langle u_i, h\rangle = 0$ for all $i \in I$.

(d) Given that $\{e^{in\theta}\}_{n\in\mathbb{Z}}$ span a dense subspace of $H = L^2(S^1, \mathbb{C})$ prove that for every $f \in H$

$$f = \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{2\pi} \langle f, e^{in\theta}\rangle e^{in\theta}.$$

You may assume that the series on the right hand side converges.