MAST30026 Metric and Hilbert Spaces

Sample exam 4

Question 1. (16 Marks)

(a) Define closure.

(b) Carefully state the theorem relating closure and limits of sequences.

(c) Carefully prove the theorem relating closure and limits of sequences.

Question 2. (20 Marks)

(a) Carefully define normed vector space.

(b) Carefully define $B(V, W)$.

(c) Prove that $B(V, W)$ is a normed vector space.

Question 3. (24 Marks)

(a) Define limit point and cluster point of a sequence in a metric space.

(b) Define Cauchy sequence and convergent sequence.

(c) Carefully state and prove a proposition to the effect that every limit point is a cluster point.

(d) Carefully state and prove a proposition to the effect that every convergent sequence is Cauchy.

Question 4. (18 Marks)

(a) Define topological space and continuous function.

(b) Define uniform space and uniformly continuous function.

(c) Prove that uniformly continuous functions are continuous.

Question 5. (20 Marks)

(a) Define topologically equivalent metric spaces.

(b) Define the standard metric d on $\mathbb{R}^2$.

(c) Let $d_2 \colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ be the metric on $\mathbb{R}^2$ given by

$$d_2((x_1, x_2), (y_1, y_2)) = |y_1 - x_1| + |y_2 - x_1|.$$

Prove that $(\mathbb{R}, d)$ and $(\mathbb{R}, d_2)$ are topologically equivalent.

Question 6. (12 Marks)

(a) Carefully state the Spectral Theorem.

(b) Give thorough examples illustrating the spectral theorem – if you were giving a lecture teaching/explaining the Spectral Theorem, what examples would you include and what points would you make about them?

Question 7. (26 Marks) Let (X, d_X) and (Y, d_Y) be metric spaces. Let

$$F = \{\text{functions } f\colon X \to Y\}, \qquad (f_1, f_2, \dots) \text{ a sequence in } F$$

and let $f\colon X \to Y$ be a function.

(a) Show that if $(f_1, f_2, \dots)$ converges uniformly to f then $(f_1, f_2, \dots)$ converges pointwise to f.

(b) Let $X = Y = \mathbb{R}_{[0,1]} = \{x \in \mathbb{R} \mid 0 \le x \le 1\}$ with metric given by $d_X(a,b) = d_Y(a,b) = |a - b|$. For $n \in \mathbb{Z}_{>0}$ let

$$\begin{matrix} f_n\colon & \mathbb{R}_{[0,1]} & \to & \mathbb{R}_{[0,1]} \\ & x & \mapsto & x^n \end{matrix} \qquad \text{and let } f\colon \mathbb{R}_{[0,1]} \to \mathbb{R}_{[0,1]}$$

be given by

$$f(x) = \begin{cases} 0, & \text{if } 0 \le x < 1, \\ 1, & \text{if } x = 1. \end{cases}$$

Carefully graph f_1, f_2, f_3, f_4 and f. Show that $(f_1, f_2, \dots)$ converges pointwise to f but does not converge uniformly to f.

Question 8. (24 Marks) Let $(a_1, a_2, \dots)$ be a bounded sequence of complex numbers. Define an operator

$$T\colon \ell^2 \to \ell^2 \qquad \text{by} \qquad T(b_1, b_2, \dots) = (0, a_1 b_1, a_2 b_2, \dots).$$

(a) Show that T is a bounded linear operator and find $\|T\|$.

(b) Compute the adjoint operator T^*.

(c) Show that if $T \neq 0$ then $T^*T \neq TT^*$.

(d) Find the eigenvalues of T^*.