21 Problem List: Spaces

21.1 The Cauchy-Schwarz and triangle inequalities

1. (Cauchy-Schwarz and the triangle inequality) Let $(V, \langle,\rangle)$ be a positive definite inner product space. The *length norm* on V is the function

$$\begin{matrix} V & \to & \mathbb{R}_{\ge 0} \\ v & \mapsto & \|v\| \end{matrix} \qquad \text{given by} \qquad \|v\|^2 = \langle v, v\rangle.$$

   Show that

   (a) If $x, y \in V$ then $|\langle x, y\rangle| \le \|x\| \cdot \|y\|$.
   (b) If $x, y \in V$ then $\|x + y\| \le \|x\| + \|y\|$.

2. (Pythagorean theorem) Let $(V, \langle,\rangle)$ be a positive definite inner product space. The *length norm* on V is the function

$$\begin{matrix} V & \to & \mathbb{R}_{\ge 0} \\ v & \mapsto & \|v\| \end{matrix} \qquad \text{given by} \qquad \|v\|^2 = \langle v, v\rangle.$$

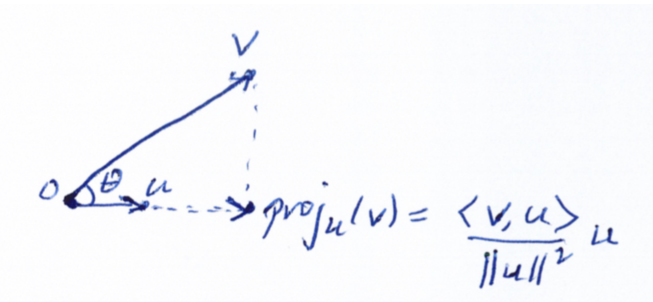
   Show that

$$\text{if } x, y \in V \text{ and } \langle x, y\rangle = 0 \qquad \text{then} \qquad \|x\|^2 + \|y\|^2 = \|x + y\|^2.$$

3. (angles and projections) Let $(V, \langle,\rangle)$ be a inner product space and let $u, v \in V$.
   The *angle between v and u* is $\theta \in [0, 2\pi)$ defined by

$$\cos(\theta) = \frac{\langle v, u\rangle}{\|v\|\,\|u\|} \qquad \text{and} \qquad \mathrm{proj}_u(v) = \left\langle v, \tfrac{u}{\|u\|}\right\rangle \frac{u}{\|u\|}.$$

   is the *orthogonal projection of v onto u*.



   (a) Use the Cauchy-Schwarz inequality to show that $0 \le \cos(\theta) < 1$ and show that $\|\mathrm{proj}_u(v)\| = \cos(\theta) \cdot \|v\|$.
   (b) Let W be a finite dimensional subspace of V and let $\{u_1, \ldots, u_k\}$ be an orthonormal basis of W. The *orthogonal projection of v onto the subspace W* is

$$\mathrm{proj}_W(v) = \langle v, u_1\rangle u_1 + \cdots + \langle v, u_k\rangle u_k.$$

   Show that $\mathrm{proj}_W(v)$ is independent of the choice of orthonormal basis.

4. Let $(V, \langle,\rangle)$ be a positive definite inner product space. The *length norm* on V is the function

$$\begin{matrix} V & \to & \mathbb{R}_{\geq 0} \\ v & \mapsto & \|v\| \end{matrix} \qquad \text{given by} \qquad \|v\|^2 = \langle v, v\rangle.$$

   (a) (The Cauchy-Schwarz inequality) Show that if $x, y \in V$ then $|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$.
   (b) (The triangle inequality) Show that if $x, y \in V$ then $\|x + y\| \leq \|x\| + \|y\|$.
   (c) (The Pythagorean theorem) Show that

$$\text{if } x, y \in V \text{ and } \langle x, y\rangle = 0 \qquad \text{then} \qquad \|x\|^2 + \|y\|^2 = \|x + y\|^2.$$

   (d) (The parallelogram law) Show that

$$\text{if } x, y \in V \text{ then } \quad \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

   (e) Show that if $(V, \|\ \|)$ is a normed vector space over $\mathbb{R}$ such that $\|\ \|: V \to \mathbb{R}_{\geq 0}$ satisfies

$$\text{if } x, y \in V \text{ then } \quad \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2,$$

   then $(V, \langle,\rangle)$ with $\langle,\rangle: V \times V \to \mathbb{R}$ given by

$$\langle x, y\rangle = \tfrac{1}{2}(\|x + y\|^2 - \|x\|^2 - \|y\|^2) = \tfrac{1}{4}(\|x + y\|^2 - \|x - y\|^2)$$

   is a positive definite symmetric inner product space such that $\|v\|^2 = \langle v, v\rangle$. To prove that $\langle x_1 + x_2, y\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle$, first establish the identity

$$\|x_1 + x_2 + y\| = \|x_1\|^2 + \|x_2\|^2 + \|x_1 + y\|^2 + \|x_2 + y\|^2 - \tfrac{1}{2}\|x_1 + y - x_2\|^2 - \tfrac{1}{2}\|x_2 + y - x_1\|^2.$$

   To prove that $\langle cx, y\rangle = \lambda cx, y\rangle$, first show that this identity holds when $c \in \mathbb{Z}$, then for $c \in \mathbb{Q}$, and finally by continuity for every $c \in \mathbb{R}$.
   (f) Show that if $(V, \|\ \|)$ is a normed vector space over $\mathbb{C}$ and $\|\ \|: V \to \mathbb{R}_{\geq 0}$ satisfies

$$\text{if } x, y \in V \text{ then } \quad \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2,$$

   then $(V, \langle,\rangle)$ with $\langle,\rangle: V \times V \to \mathbb{C}$ given by

$$\langle x, y\rangle = \tfrac{1}{4}(\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2)$$

   is a positive definite Hermitian inner product space such that $\|v\|^2 = \langle v, v\rangle$.

21.2 Relating types of spaces

1. (positive definite inner product spaces are normed vector spaces) Let $(V, \langle,\rangle)$ be a positive definite inner product space. The *length norm* on V is the function

$$\begin{matrix} V & \to & \mathbb{R}_{\geq 0} \\ v & \mapsto & \|v\| \end{matrix} \qquad \text{given by} \qquad \|v\|^2 = \langle v, v\rangle.$$

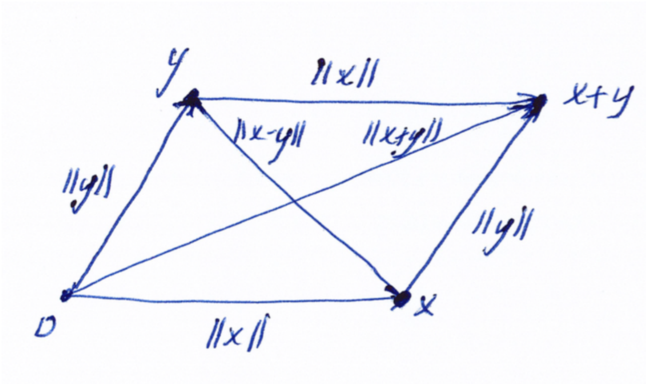
   Show that $(V, \|\ \|)$ is a normed vector space.

2. (inner product spaces from normed vector spaces: the parallelogram law)

(a) Let $(V, \langle, \rangle)$ be a inner product space and let $\| \|: V \to \mathbb{R}_{\geq 0}$ be given by $\|v\|^2 = \langle v, v\rangle$. Show that

$$\text{if } x, y \in V \text{ then} \quad \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

(the sum of the squared lengths of the edges is the sum of the squared lengths of the daigonals).



(b) Show that if $(V, \| \|)$ is a normed vector space over $\mathbb{R}$ such that $\| \|: V \to \mathbb{R}_{\geq 0}$ satisfies

$$\text{if } x, y \in V \text{ then} \quad \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2,$$

then $(V, \langle, \rangle)$ with $\langle, \rangle: V \times V \to \mathbb{K}$ given by

$$\langle x, y\rangle = \tfrac{1}{2}(\|x+y\|^2 - \|x\|^2 - \|y\|^2) = \tfrac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$$

is a positive definite symmetric inner product space such that $\|v\|^2 = \langle v, v\rangle$. To prove that $\langle x_1 + x_2, y\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle$, first establish the identity

$$\|x_1 + x_2 + y\| = \|x_1\|^2 + \|x_2\|^2 + \|x_1+y\|^2 + \|x_2+y\|^2 - \tfrac{1}{2}\|x_1 + y - x_2\|^2 - \tfrac{1}{2}\|x_2 + y - x_1\|^2.$$

To prove that $\langle cx, y\rangle = c\langle x, y\rangle$, first show that this identity holds when $c \in \mathbb{Z}$, then for $c \in \mathbb{Q}$, and finally by continuity for every $c \in \mathbb{R}$. (See [Bre, Ch. 5 Ex. 3].)

(c) Show that if $(V, \| \|)$ is a normed vector space over $\mathbb{C}$ such that $\| \|: V \to \mathbb{R}_{\geq 0}$ satisfies

$$\text{if } x, y \in V \text{ then} \quad \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2,$$

then $(V, \langle, \rangle)$ with $\langle, \rangle: V \times V \to \mathbb{C}$ given by

$$\langle x, y\rangle = \tfrac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2)$$

is a positive definite Hermitian inner product space such that $\|v\|^2 = \langle v, v\rangle$. (See [Ru, Ch. 4 Ex. 11].)

3. (normed vector spaces are metric spaces) Let $(V, \| \|)$ be a normed vector space. The *norm metric* on V is the function

$$d: V \times V \to \mathbb{R}_{\geq 0} \qquad \text{given by} \qquad d(x, y) = \|x - y\|.$$

Show that (V, d) is a metric space.

4. (uniformity of a pseudometric) Let X be a set. A *pseudometric on* X is a function $f\colon X\times X\to \mathbb{R}_{\ge 0}\cup\{\infty\}$ such that

   (a) If $x\in X$ then $d(x,x)=0$,
   (b) If $x,y\in X$ then $d(x,y)=d(y,x)$,
   (c) If $x,y,z\in X$ then $d(x,y)\le d(x,z)+d(z,y)$.

   Show that the sets

   $$B_\epsilon=\{(x,y)\in X\times X \mid d(x,y)<\epsilon\},\qquad \text{for } \epsilon\in\mathbb{R}_{>0},$$

   generate a uniformity $\mathcal{X}_d$ on X. (See [Bou, Top. Ch. IX §1 no. 2].)

5. (every uniformity comes from a family of pseudometrics) Let $(X,\mathcal{X})$ be a uniform space. Show that there exists a set $\mathcal{D}$ of pseudometrics on X such that $\mathcal{X}$ is the least upper bound of the set $\{\mathcal{X}_d \mid d\in\mathcal{D}\}$ of uniformities $\mathcal{X}_d$ defined by the pseudometrics $d\in\mathcal{D}$. (See [Bou, Top. Ch. IX §1 no. 4 Theorem 1].)

6. (The neighborhood filter of a uniform space) Let $(X,\mathcal{X})$ be a uniform space. Let $x\in X$ and let $\mathcal{N}(x)$ be the neighborhood filter of x. Show that

   $$\mathcal{N}(x)=\{B_V(x)\mid V\in\mathcal{X}\}.$$

7. (The uniform space topology is a topology) Let $(X,\mathcal{X})$ be a uniform space. Let

   $$B_V(x)=\{y\in X\mid (x,y)\in V\}\quad \text{for } V\in\mathcal{X} \text{ and } x\in X,\quad \text{and let}$$
   $$\mathcal{N}(x)=\{B_V(x)\mid V\in\mathcal{X}\}\quad \text{for } x\in X.$$

   (a) Show that $\mathcal{T}=\{U\subseteq X\mid \text{if } x\in U \text{ then } U\in\mathcal{N}(x)\}$ is a topology on X.
   (b) Show that if $\mathcal{U}$ is a topology on X and $\mathcal{U}\supseteq\{B_V(x)\mid V\in\mathcal{X}\}$ then $\mathcal{U}\supseteq\mathcal{T}$.

8. (The metric space topology is a topology) Let (X,d) be a metric space. Let

   $$B_\epsilon(x)=\{y\in X\mid d(y,x)<\epsilon\}\quad \text{for } \epsilon\in\mathbb{R}_{>0} \text{ and } x\in X.$$

   Let $\mathcal{B}=\{B_\epsilon(x)\mid \epsilon\in\mathbb{R}_{>0}, x\in X\}$.

   (a) Show that $\mathcal{T}=\{\text{unions of sets in } \mathcal{B}\}$ is a topology on X.
   (b) Show that if $\mathcal{U}$ is a topology on X and $\mathcal{U}\supseteq\mathcal{B}$ then $\mathcal{U}=\mathcal{T}$.

9. (warning on relating the metric space uniformity and the metric space topology) Let (X,d) be a metric space, $\mathcal{X}$ the metric space uniformity on X and $\mathcal{T}$ the metric space topology on X.

   a) Show that if X is discrete then $\mathcal{T}=\{\text{unions of } B_v(x)\}$ and

   $$\{B_V(x)\mid V\in\mathcal{X},\ x\in X\}=\{B_\epsilon(x)\mid \epsilon\in\mathbb{R}_{>0},\ x\in X\}.$$

(b) Show that if X is not discrete then

$$\{B_V(x) \mid V \in \mathcal{X},\ x \in X\} \quad \text{is not equal to} \quad \{B_\epsilon(x) \mid \epsilon \in \mathbb{R}_{>0},\ x \in X\}.$$

(c) Give an example to show that if X is not discrete then

$$\mathcal{T} \text{ is not equal to } \{\text{unions of } B_V(x)\}.$$

10. (Example of a topological space that is not a uniform space) Let $X = \{0,1\}$ and let $\mathcal{T} = \{\emptyset, \{0\}, X\}$. Show that $\mathcal{T}$ is a topology on X and that there does not exist a uniformity on X such that $\mathcal{T}$ is the uniform space topology on X.

11. (Example of a topological space that is not a metric space) Let $X = \{0,1\}$ and let $\mathcal{T} = \{\emptyset, \{0\}, X\}$. Show that $\mathcal{T}$ is a topology on X and that there does not exist a metric $d\colon X \times X \to \mathbb{R}_{\geq 0}$ such that $\mathcal{T}$ is the metric space topology on X. (Show that $\mathcal{T}$ is not Hausdorff.)

12. (Example of a uniform space that is not a metric space) Let $X = \{0,1\}$ and let $\mathcal{X} = \{X \times X\}$. Show that $\mathcal{X}$ is a uniformity on X and that there does not exist a metric $d\colon X \times X \to \mathbb{R}_{\geq 0}$ such that $\mathcal{X}$ is the metric space uniformity on X. (Show that the uniform space topology of X is not Hausdorff.)

13. (consistency of metric space topology, uniform space topology and metric space uniformity) Let (X,d) be a metric space and let $\mathcal{X}$ be the metric space uniformity on X. Show that the uniform space topology of $(X,\mathcal{X})$ is the same as the metric space topology on (X,d).

14. (necessary and sufficient condition for a topology to be a uniform space topology) Let $(X,\mathcal{T})$ be a topological space. Show that there exists a uniformity $\mathcal{X}$ on X such that $\mathcal{T}$ is the uniform space topology on $(X,\mathcal{X})$ if and only if $(X,\mathcal{T})$ satisfies

if $x \in X$ and V is a neighborhood of x
then there exists a continuous function $f\colon X \to [0,1]$

$$\text{with} \qquad f(x) = 0 \quad \text{and} \quad f(V^c) = \{1\}.$$

(See [Bou, Top. Ch. IX §1 no. 5 Theorem 2].)

15. (necessary conditions for a topology to be a metric space topology) Let $(X,\mathcal{T})$ be a topological space.

- $(X,\mathcal{T})$ is *Hausdorff* if X satisfies: if $x, y \in X$ and $x \neq y$ then there exist open sets U and V in X such that
$$x \in U, \qquad y \in V \qquad \text{and} \qquad U \cap V = \emptyset.$$
- $(X,\mathcal{T})$ is *normal* if X satisfies: if A and B are closed sets in X and $A \cap B = \emptyset$ then there exist open sets U and V in X such that
$$A \subseteq U, \qquad B \subseteq V \qquad \text{and} \qquad U \cap V = \emptyset.$$

- $(X, \mathcal{T})$ is *first countable* if $\mathcal{N}(a)$ is countably generated for each $a \in X$,

i.e. $(X, \mathcal{T})$ is *first countable* if X satisfies: if $a \in X$ then

$$\begin{array}{l}\text{there exist } N_1, N_2, \ldots \in \mathcal{N}(a) \text{ such that} \\ \text{if } N \in \mathcal{N}(a) \quad \text{then} \quad \text{there exists } r \in \mathbb{Z}_{>0} \text{ such that } N \supseteq N_r.\end{array}$$

Let (X, d) be a metric space and let $\mathcal{T}$ be the metric space topology on X. Show that

(a) $(X, \mathcal{T})$ is Hausdorff,
(b) $(X, \mathcal{T})$ is normal,
(c) $(X, \mathcal{T})$ is first countable.

16. (sufficient condition for a topology to be a metric space topology) A topological space $(X, \mathcal{T})$ is *regular* if $(X, \mathcal{T})$ is Hausdorff and

$$\text{if } x \in X \quad \text{then} \quad \begin{array}{l}\{C \subseteq X \mid C \text{ is closed and } x \in C\} \\ \text{is a fundamental system of neighborhoods of } x.\end{array}$$

Let $(X, \mathcal{T})$ be a topological space. Show that

$$\text{if } (X, \mathcal{T}) \text{ is regular} \quad \text{and} \quad \mathcal{T} \text{ has a countable base}$$

then there exists a metric $d \colon X \times X \to \mathbb{R}_{\geq 0}$ on X such that $\mathcal{T}$ is the metric space topology of (X, d). (See [Bou, Top. Ch. IX §4 Ex. 22].)

17. (necessary and sufficient condition for a topology to be a metric space topology) Let $(X, \mathcal{T})$ be a topological space. There exists a metric $d \colon X \times X \to \mathbb{R}_{\geq 0}$ on X such that $\mathcal{T}$ is the metric space topology of (X, d) if and only if

(a) $(X, \mathcal{T})$ is regular and
(b) there exists a sequence $(\mathcal{B}_1, \mathcal{B}_2, \ldots)$ of locally finite families of open subsets of X such that $\mathcal{B} = \bigcup_{n \in \mathbb{Z}_{>0}} \mathcal{B}_n$ is a base of the topology $\mathcal{T}$.

(See [Bou, Top. Ch. IX §4 Ex. 22].)

18. (necessary and sufficient condition for a uniformity to be a metric space uniformity) Let $(X, \mathcal{X})$ be a uniform space and let $\mathcal{T}$ be the uniform space topology of $(X, \mathcal{X})$.

$$\text{There exists a metric} \quad d \colon X \times X \to \mathbb{R}_{\geq 0}$$

on X such that $\mathcal{X}$ is the metric space uniformity of (X, d) if and only if

(a) $(X, \mathcal{T})$ is Hausdorff and
(b) there exists a countable subset $\mathcal{B}$ of $\mathcal{X}$ such that

$$\mathcal{X} = \{V \subseteq X \times X \mid V \text{ contains a set in } \mathcal{B}\}.$$

(See [Bou, Top. Ch. IX §5 no. 4 Theorem 1].)

21.3 The poset of topologies

1. (union generating set of a topology) Let $(X, \mathcal{T})$ be a topological space.

   A *union generating set*, or *base, of* $\mathcal{T}$ is a collection $\mathcal{B}$ of subsets of X such that

   $$\mathcal{T} = \{\text{unions of sets in } \mathcal{B}\}.$$

   Show that $\mathcal{B}$ is a base of the topology $\mathcal{T}$ if and only if $\mathcal{B}$ satisfies

   (a) (intersection covering) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ then

   $$\text{there exists } B \in \mathcal{B} \text{ such that} \quad x \in B \text{ and } B \subseteq B_1 \cap B_2.$$

   (b) (cover) $\bigcup_{B\in\mathcal{B}} B = X$.

2. (The metric space topology) Let (X, d) be a metric space. Show that

   $$\mathcal{B} = \{B_\epsilon(x) \mid \epsilon \in \mathbb{R}_{>0}, x \in X\}$$

   is a union generating set of the metric space topology on X.

3. (The discrete topology) Let X be a set. The *power set of* X, or the *discrete topology on* X, is

   $$\text{the set} \quad \mathcal{P}(X) = \{A \subseteq X\} \quad \text{of all subsets of } X.$$

   Show that $\mathcal{P}(X)$ is a topology on X.

4. (The cofinite topology) A topological space $(X, \mathcal{T})$ is *Hausdorff* if it satisfies: if $x, y \in X$ and $x \neq y$ then there exist open sets U and V in X such that

   $$x \in U, \qquad y \in V \qquad \text{and} \qquad U \cap V = \emptyset.$$

   A topological space $(X, \mathcal{T})$ is *normal* if it satisfies: if A and B are closed sets in X and $A \cap B = \emptyset$ then there exist open sets U and V in X such that

   $$A \subseteq U, \qquad B \subseteq V \qquad \text{and} \qquad U \cap V = \emptyset.$$

   A topological space $(X, \mathcal{T})$ is *first countable* if it satisfies

   $$\text{if } a \in X \quad \text{then} \quad \text{there exists} \quad \begin{matrix} \text{a countable collection of neighborhoods of } a \\ \text{which generates the neighborhood filter } \mathcal{N}(a) \text{ of } a. \end{matrix}$$

   In other words, a topological space $(X, \mathcal{T})$ is first countable if it satisfies: if $a \in X$ then there exists $N_1, N_2, \ldots \in \mathcal{N}(a)$ such that if $N \in \mathcal{N}(a)$ then there exists $i \in \mathbb{Z}_{>0}$ such that $N \supseteq N_i$.

   Let X be a set and let $\mathcal{T}$ be the topology such that the closed sets are the finite subsets of X.

   (a) Show that if X is finite then $\mathcal{T}$ is the discrete topology on X.
   (b) Show that if X is infinite then $(X, \mathcal{T})$ is not Hausdorff and not normal.
   (c) Show that if X is uncountable then $(X, \mathcal{T})$ is not first countable.

5. (The poset of topologies on X) Let X be a set and let $\mathcal{P}(X) = \{A \subseteq X\}$ be the power set of X. Show that $\subseteq$ is a partial order on the set $\mathcal{P}(\mathcal{P}(X))$ of all subsets of $\mathcal{P}(X)$. Let $\mathcal{T}(\mathcal{P}(X)$ be the set of all topologies on X. Show that $\mathcal{T}(\mathcal{P}(X))$ is a subposet of $\mathcal{P}(\mathcal{P}(X)))$.

6. (topologies and uniformities on a 2 element set) Let X be a set with 2 elements. Show that there are four possible topologies on X and two possible uniformities on X. Determine the uniform space topology of each uniformity on X.

7. (topologies on a 3 element set) Let X be a set with 3 elements. Determine all possible topologies on X.

8. (the order topology) Give an example of a poset X such that the collection $\mathcal{T} = \{\text{unions of open intervals}\}$ is not a topology. (Instead one should take the topology generated by the set of open intervals in X.) See [Bou] Top. Ch. I §1 Ex. 2 and §2 Ex. 5].

21.4 Topologically equivalent metric spaces

1. (Lipschitz equivalence implies topological equivalence) Let X be a set and let

$$d_1\colon X \times X \to \mathbb{R}_{\geq 0} \qquad \text{and} \qquad d_2\colon X \times X \to \mathbb{R}_{\geq 0} \qquad \text{be metrics on } X.$$

The metrics d_1 and d_2 are *topologically equivalent* if

$$\text{the metric space topology on } (X, d_1) \text{ and on } (X, d_2) \quad \text{are the same.}$$

The metrics d_1 and d_2 are *Lipschitz equivalent* if there exist $c_1, c_2 \in \mathbb{R}_{>0}$ such that

$$\text{if } x, y \in X \quad \text{then} \quad c_1 d_2(x, y) \leq d_1(x, y) \leq c_2 d_1(x, y).$$

Show that if d_1 and d_2 are Lipschitz equivalent then d_1 and d_2 are topologically equivalent.

2. (every metric space is topologically equivalent to a bounded metric space) A metric space (X, d) is *bounded* if it satisfies

$$\text{there exists } M \in \mathbb{R}_{\geq 0} \text{ such that} \quad \text{if } x_1, x_2 \in X \text{ then } d(x_1, x_2) < M.$$

Let (X, d) be a metric space and define $b\colon X \times X \to \mathbb{R}_{\geq 0}$ by

$$b(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

(a) Show that $b\colon X \times X \to \mathbb{R}_{\geq 0}$ is a metric on X.
(b) Show that the metric space topology of (X, b) and the metric space topology on (X, d) are the same.
(c) Show that (X, b) is a bounded metric space.

3. (boundedness is not a topological property) A metric space (X, d) is *bounded* if it satisfies

$$\text{there exists } M \in \mathbb{R}_{>0} \text{ such that} \quad \text{if } x_1, x_2 \in X \text{ then } d(x_1, x_2) < M.$$

   (a) Let $X = \mathbb{R}$ and let $d\colon X \times X \to \mathbb{R}_{\geq 0}$ and $b\colon X \times X \to \mathbb{R}_{\geq 0}$ be the metrics on $\mathbb{R}$ given by

$$d(x, y) = |x - y| \qquad \text{and} \qquad b(x, y) = \frac{|x - y|}{1 + |x - y|}.$$

   Show that (X, d) and (X, b) have the same topology, that (X, d) is unbounded, and (X, b) is bounded.

   (b) Let $X = \mathbb{R}^2$ and let $d\colon X \times X \to \mathbb{R}_{\geq 0}$ and $b\colon X \times X \to \mathbb{R}_{\geq 0}$ be the metrics on $\mathbb{R}$ given by

$$d(x, y) = |x - y| \qquad \text{and} \qquad b(x, y) = \frac{|x - y|}{1 + |x - y|}.$$

   Draw pictures of the open balls $B_{\frac{1}{2}}(0)$, $B_{\frac{3}{4}}(0)$, $B_{\frac{9}{10}}(0)$ and $B_{\frac{99}{100}}(0)$ for the metric $b\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$.

4. Let (X, d) be a metric space. Show that the metric $d'\colon X \times X \to \mathbb{R}$ given by

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

   is topologically equivalent to d.

5. Let (X, d) be a metric space. Show that (X, d') is a bounded metric space, where

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

6. Give an example of X and two metrics d and d' on X such that d is topologically equivalent to d' and (X, d) is not bounded and (X, d') is bounded.

7. Let $(X_1, d_1), \ldots, (X_\ell, d_\ell)$ be metric spaces and let $(X_1 \times \cdots \times X_\ell, d)$ be the product metric space. Let $\sigma\colon (X_1 \times \cdots \times X_\ell) \times (X_1 \times \cdots \times X_\ell) \to \mathbb{R}$ be given by

$$\sigma(x, y) = \max\{d_i(x_i, y_i) \mid 1 \leq i \leq \ell\}.$$

   Show that σ is a metric on $X_1 \times \cdots \times X_\ell$ and d is topologically equivalent to σ.

8. Let $(X_1, d_1), \ldots, (X_\ell, d_\ell)$ be metric spaces and let $(X_1 \times \cdots \times X_\ell, d)$ be the product metric space. Let $\rho\colon (X_1 \times \cdots \times X_\ell) \times (X_1 \times \cdots \times X_\ell) \to \mathbb{R}$ be given by

$$\rho(x, y) = \left(\sum_{i=1}^{\ell} d_i(x_i, y_i)^2 \right)^{\frac{1}{2}}.$$

   Show that ρ is a metric on $X_1 \times \cdots \times X_\ell$ and d is topologically equivalent to ρ.

9. Let X be a set and let d and d' be metrics on X. Show that d and d' are topolgically equivalent if d and d' satisfy the condition

$$\text{if } x, y \in X \quad \text{then} \quad \text{there exist } k, k' \in \mathbb{R} \quad \text{such that} \quad d(x,y) \le kd'(x,y) \le k'd(x,y).$$

10. Let X be a set. Metrics d and $\overline{d}$ defined on X are *Lipschitz equivalent* if there exist $m, M \in \mathbb{R}_{>0}$ such that

$$\text{if } x, y \in X \quad \text{then} \quad md(x,y) \le \overline{d}(x,y) \le Md(x,y)$$

   (a) Show that if d and $\overline{d}$ are Lipschitz equivalent, then they are toplogically equivalent.
   (b) Give an example of X and two topologically equivalent metrics on X which are not Lipschitz equivalent.
   (c) For $p \ge 1$ and $x, y \in \mathbb{R}^n$, the l^p metric is defined by

$$d_p(x,y) = \Big(\sum_{i=1}^{n} |x_i - y_i|^p\Big)^{1/p} = \|x - y\|_p.$$

   Show that if $p, q \ge 1$, then d_p and d_q are Lipschitz equivalent. (Hint: compare these with $d_\infty(x,y) = \max(|x_1 - y_1|, \ldots, |x_n - y_n|)$.)

11. (limit definition of topological equivalence) PUT THIS IN

21.5 Favourite examples of metric and normed spaces

1. (example of a nonHausdorff space) Let $X = \{(x,1) \mid x \in \mathbb{R}\} \cup \{(0,2)\}$ with

$$d((x_1,y_1),(x_2,y_2))| = |x_1 - x_2| \qquad \text{and topology} \quad \mathcal{T} = \{\text{unions of sets in } \mathcal{B}\},$$

   where $\mathcal{B} = \{B_\epsilon(x,y) \mid \epsilon \in \mathbb{R}_{>0}, (x,y) \in X\}$ and

$$B_\epsilon(x,y) = \{(a,b) \in X \mid d((a,b),(x,y)) < \epsilon.$$

   Show that X is a non Hausdorff topological space.

2. (the two point space) Let X be a set.
   (a) Carefully define a "topology on X" and a "uniformity on X".
   (b) Let (X, d) be a metric space. Carefully define the "metric space topology on X" and the "metric space uniformity on X".
   (c) Determine all the topologies on the set $X = \{0, 1\}$.
   (d) Determine all the uniformities on $X = \{0, 1\}$.
   (e) For each of the uniformities you gave in part (d), compute the uniform space topology.

3. Define the standard metric on $\mathbb{C}$ and show that $\mathbb{C}$, with this metric, is a metric space.

4. Let d be the standard metric on $\mathbb{C}$. Show that $\mathbb{R}$ is a metric subspace of $(\mathbb{C}, d)$.

5. Let X be a set. Define the standard metric on X and show that X, with this metric, is a metric space.

6. Let $(X_1, d_1), \ldots, (X_n, d_n)$ be metric spaces. Define the product metric d on $X_1 \times X_2 \times \cdots \times X_n$ and show that $(X_1 \times \cdots \times X_n, d)$ is a metric space.

7. Let $(X, \| \ \|)$ be a normed vector space. Define the standard metric on X and show that X, with this metric, is a metric space.

8. Define the standard metric on $\mathbb{R}^n$ and show that $\mathbb{R}^n$, with this metric, is a metric space.

9. Define the standard norm on $\mathbb{R}^n$ and show that $\mathbb{R}^n$, with this norm, is a normed vector space.

10. Define the norm $\| \ \|_p$ on $\mathbb{R}^n$ and show that $(\mathbb{R}^n, \| \ \|_p)$ is a normed vector space.

11. Let X be a nonempty set. Define the set of bounded functions $B(X, \mathbb{R})$ and the sup norm on $B(X, \mathbb{R})$. Show that $B(X, \mathbb{R})$, with this norm, is a normed vector space.

12. Let $a, b \in \mathbb{R}$ with $a < b$. Define the set of continuous functions $C([a, b], \mathbb{R})$ and the L^1-norm on $C([a, b], \mathbb{R})$. Show that $C([a, b], \mathbb{R})$, with this norm, is a normed vector space.

13. Let $a, b \in \mathbb{R}$ with $a < b$. Show that the set $C_{\mathrm{bd}}([a, b]), \mathbb{R})$ of bounded continuous functions is a metric subspace of $C([a, b], \mathbb{R})$ with the L^1-norm.

14. Let (X, d) be a metric space. Define the metric space topology on X and show that it is a topology on X.

15. Let X be a set and let d be the discrete metric on X. Determine which subsets of X are in the metric space topology on X.

16. Give two metrics d and d' on $\mathbb{R}$ such that $\mathbb{Q}$ is open in the metric space topology on $(\mathbb{R}, d)$ and $\mathbb{Q}$ is not open in the metric space topology on $(\mathbb{R}, d')$.

17. Let $X = \{0, 1\}$ and let $\mathcal{T} = \{\emptyset, X, \{0\}\}$.

    (a) Show that $\mathcal{T}$ is a topology on X.

    (b) Show that there does not exist a metric $d \colon X \times X \to \mathbb{R}_{\geq 0}$ such that $\mathcal{T}$ is the metric space topology of (X, d).

18. Check if the following functions are metrics on X.

    (a) $X = \mathbb{R}$ and $d(x, y) = |x^2 - y^2|$.
    (b) $X = (-\infty, 0]$ and $d(x, y) = |x^2 - y^2|$.
    (c) $X = \mathbb{R}$ and $d(x, y) = |\arctan x - \arctan y|$.

19. (The French railroad metric) Let $X = \mathbb{R}^2$ and let d be the usual metric. Let $\mathbf{0} = (0, 0)$ and define

$$d_{\mathbf{0}}(x, y) = \begin{cases} 0, & \text{if } x = y; \\ d(x, \mathbf{0}) + d(\mathbf{0}, y), & \text{if } x \neq y. \end{cases}$$

    Verify that d_0 is a metric on X. (Paris is at the origin $\mathbf{0}$.)

20. Let $X = \mathbb{R}^2$. For $x = (x_1, x_2)$ and $y = (y_1, y_2)$ define

$$d(x, y) = \begin{cases} 1/2, & \text{if } x_1 = y_1 \text{ and } x_2 \neq y_2 \text{ or if } x_1 \neq y_1 \text{ and } x_2 = y_2; \\ 1, & \text{if } x_1 \neq y_1 \text{ and } x_2 \neq y_2; \\ 0, & \text{otherwise.} \end{cases}$$

    Verify that d is a metric and that two congruent rectangles, one with base parallel to the x-axis and the other at 45° to the x-axis, have different "area" if d is used to measure the length of sides.

21. Let (X, d) be a metric space. Let $f\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a function such that

    (a) If $0 \leq a < b$ then $f(a) \leq f(b)$,
    (b) $f(x) = 0$ if and only if $x = 0$, and
    (c) $f(a + b) \leq f(a) + f(b)$.

    Define $d_f\colon X \times X \to \mathbb{R}_{\geq 0}$ by

$$d_f(x, y) = f(d(x, y)).$$

    Show that d_f is a metric. Let $k \in \mathbb{R}_{>0}$ and $\alpha \in \mathbb{R}_{(0,1]}$. Show that the functions

$$f(t) = kt, \qquad f(t) = t^\alpha \qquad \text{and} \qquad f(t) = \frac{t}{1+t},$$

    have properties (a), (b) and (c).

22. (the p-adic metric) Let X be a set. An *ultrametric on* X is a function $d\colon X \times X \to \mathbb{R}_{\geq 0}$ such that

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}.$$

    Let p be a prime number. Define the *p-adic absolute value function* $|\ |_p\colon \mathbb{Q} \to \mathbb{Q}_{\geq 0}$ by

$$|x|_p = \begin{cases} 0, & \text{if } x = 0, \\ p^{-k}, & \text{if } x = p^k \cdot \frac{m}{n}, \text{ with } m, n \in \mathbb{Z}_{\neq 0} \text{ not divisible by } p. \end{cases}$$

    (a) Show that if X is a set and d is an ultrametric on X then d is a metric on X.

(b) Show that if $x, y \in \mathbb{Q}$ then
$$|x+y|_p \le \max\{|x|_p, |y|_p\}.$$
(c) Show that $d_p\colon \mathbb{Q}\times\mathbb{Q}\to\mathbb{R}_{\ge 0}$ given by
$$d_p(x,y) = |x-y|_p \qquad \text{is an ultrametric on } \mathbb{Q}.$$

23. (product metrics) Let $(X_1, d_1), \ldots, (X_n, d_n)$ be metric spaces and let $X = X_1 \times \cdots \times X_n$. Define
$$d(x,y) = (d_1(x_1,y_1) + \cdots + d_n(x_n,y_n))^{\frac{1}{2}},$$
$$\overline{d}(x,y) = \max\{d_1(x_1,y_1), \ldots, d_n(x_n,y_n)\},$$
where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n) \in X$. Verify that d and $\overline{d}$ are metrics on X.

24. (Polynomials of degree $\le n$ as a normed vector space) Fix a positive integer n. Denote by
$$\mathcal{P}_n = \{p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \mid a_1, \ldots, a_n \in \mathbb{R}\}.$$
For $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathcal{P}_n$ set
$$\|p\| = \max\{|a_0|, |a_1|, \ldots, |a_n|\}.$$
Verify that $\|\ \|$ is a norm on $\mathcal{P}_n$.

25. (An infinite product space) Let $(X_1, d_2), (X_2 d_2), \ldots,$ be a sequence of metric spaces. Let
$$X = \left(\prod_{n\in\mathbb{Z}_{>0}} X_n\right) = \{\ x = (x_1, x_2, \ldots) \mid x_n \in X_n\}.$$
For $x, y \in X$ let
$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n}\left(\frac{d_n(x_n,y_n)}{1+d_n(x_n,y_n)}\right).$$
Show that (X, d) is a metric space.

26. (the shape of product metrics) Sketch the open ball $B_1(0)$ in each of the metric spaces $(\mathbb{R}^3, d_1)$, $(\mathbb{R}^3, d_2)$, and $(\mathbb{R}^3, d_\infty)$, where
$$d_1(x,y) = |x_1-y_1| + |x_2-y_2| + |x_3-y_3|$$
$$d_2(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2}$$
$$d_\infty(x,y) = \max\{|x_1-y_1|, |x_2-y_2|, |x_3-y_3|\}.$$
for $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3) \in \mathbb{R}^3$.

27. (a metric on the positive integers) Define $d\colon \mathbb{Z}_{>0}\times\mathbb{Z}_{>0}\to\mathbb{R}_{\ge 0}$ by
$$d(n,m) = \left|\tfrac{1}{n} - \tfrac{1}{m}\right|.$$
(a) Show that d is a metric.
(b) Let $P \subseteq \mathbb{Z}_{>0}$ be the set of positive even numbers. Find $\mathrm{diam}(P)$ and $\mathrm{diam}(\mathbb{Z}_{>0}\setminus P)$ in $(\mathbb{Z}_{>0}, d)$.
(c) Let $n \in \mathbb{Z}_{>0}$. Find all elements of $B_{\frac{1}{2n}}(2n)$ and $B_{\frac{1}{2n}}(n)$.

21.6 Distances and diameters

1. Let X be a non-empty set and let $d : X \times X \to \mathbb{R}$ be a function such that

   (i) $d(x, y) = 0$ if and only if $x = y$,
   (ii) if $x, y, z \in X$ then $d(x, y) \le d(x, z) + d(y, z)$.

   Prove that d is a metric on X and show that $d(y, z) \ge |d(x, y) - d(x, z)|$.

2. Let A and B be bounded subsets of a metric space (X, d) such that $A \cap B \neq \emptyset$. Show that

$$\mathrm{diam}(A \cup B) \le \mathrm{diam}(A) + \mathrm{diam}(B).$$

   What can you say if A and B are disjoint?

3. (diameter of an open ball) Let (X, d) be a metric space. Let $x_0 \in X$ and let $r \in \mathbb{R}_{>0}$.

   (a) Show that $\mathrm{diam}(B_r(x_0)) \le 2r$.
   (b) Give an example showing that the strict inequality is possible.

4. Let (X, d) be a metric space.

   (a) Prove that if $x, x', y, y' \in X$ then

$$|d(x, y) - d(x', y')| \le d(x, x') + d(y, y').$$

   (b) Let A be a non-empty compact subset of X. Prove that there exist $a, b \in A$ such that

$$d(a, b) = \sup\{d(x, y) \mid x, y \in A\}.$$