25 Problem list: Compactness

25.1 Relating types of compactness

1. (cover compact implies sequentially compact) Let (X, d) be a metric space and let $A \subseteq X$. Show that if A is cover compact then A is sequentially compact.

2. (sequentially compact implies cover compact) Let (X, d) be a metric space and let $A \subseteq X$. Show that if A is sequentially compact then A is cover compact.

3. (sequentially compact implies Cauchy compact) Let (X, d) be a metric space and let $A \subseteq X$. Show that if A is sequentially compact then A is Cauchy compact.

4. (cover compact implies ball compact) Let (X, d) be a metric space and let $A \subseteq X$. Show that if A is cover compact then A is ball compact.

5. (ball compact implies bounded) Let (X, d) be a metric space and let $A \subseteq X$. Show that if A is ball compact then A is bounded.

6. (sequentially compact implies Cauchy compact) Let (X, d) be a metric space and let $A \subseteq X$. Show that if A is Cauchy compact then A is closed.

7. (ball compact does not imply closed) Let $A = (0, 1) \subseteq \mathbb{R}$ with the standard metric on $\mathbb{R}$. Show that A is ball compact and not closed.

8. (ball compact does not imply cover compact) Let $A = (0, 1) \subseteq \mathbb{R}$ with the standard metric on $\mathbb{R}$. Show that A is ball compact and not cover compact.

9. (ball compact does not imply Cauchy compact) Let $A = (0, 1) \subseteq \mathbb{R}$ with the standard metric on $\mathbb{R}$. Show that A is ball compact and not Cauchy compact.

10. (bounded does not imply ball compact) Let $X = \mathbb{R}$ with metric given by $d(x, y) = \min\{|x-y|, 1\}$ and let $A = X$. Show that A is bounded but not ball compact.

11. (closed does not imply Cauchy compact) Let $X = \mathbb{R}_{(0,1)} = \{x \in \mathbb{R} \mid 0 < x < 1\}$ with metric given by $d(x, y) = |x - y|$ and let $A = X$. Show that A is closed in X but not Cauchy compact.

12. (Cauchy compact does not imply bounded) Let $X = \mathbb{R}$ with metric given by $d(x, y) = |x - y|$ and let $A = X$. Show that A is Cauchy compact but not bounded.

13. (Cauchy compact does not imply cover compact) Let $X = \mathbb{R}$ with metric given by $d(x, y) = |x-y|$ and let $A = X$. Show that A is Cauchy compact but not cover compact.

14. (Cauchy compact does not imply ball compact) Let $X = \mathbb{R}$ with metric given by $d(x, y) = |x - y|$ and let $A = X$. Show that A is Cauchy compact but not ball compact.

15. (ball compact+Cauchy compact implies cover compact) Let (X, d) be a metric space and let $A \subseteq X$. Show that if A is ball compact and Cauchy compact if and only if A is cover compact.

16. (In $\mathbb{R}^n$ closed and bounded implies cover compact) Let $X = \mathbb{R}^n$ with the standard metric and let $A \subseteq X$. Show that A closed and bounded if and only if A is cover compact.

17. (In $\mathbb{R}^n$ closed implies Cauchy compact) Let $X = \mathbb{R}^n$ with the standard metric and let $A \subseteq X$. Show that if A is closed in X then A is Cauchy compact.

18. (closed subsets of Cauchy compact spaces are Cauchy compact) Let (X, d) be a Cauchy compact metric space and let $A \subseteq X$. Show that if A is closed in X then A is Cauchy compact.

19. (bounded subsets of ball compact spaces are ball compact) Let (X, d) be a ball compact metric space and let $A \subseteq X$. Show that if A is bounded then A is ball compact.

20. (closed subsets of cover compact spaces are cover compact) Let (X, d) be a cover compact metric space and let $A \subseteq X$. Show that if A is closed in X then A is cover compact.

21. (compact subsets of Hausdorff topological spaces are closed) Let $(X, \mathcal{T})$ be a Hausdorff topological space and let K be a compact subset of X. Let $x \in K^c$. Since X is Hausdorff, for each $y \in K$ there exist $U_{xy} \in \mathcal{T}$ and $V_{xy} \in \mathcal{T}$ such that

$$U_{xy} \cap V_{xy} = \emptyset \qquad \text{and then} \qquad \{V_{xy} \mid y \in K\} \quad \text{is an open cover of } K.$$

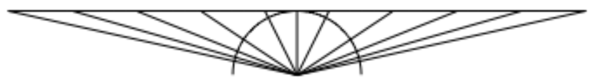
Since K is compact there exists a finite subcover $\{V_{xy_1}, V_{xy_2}, \ldots, V_{xy_\ell}\}$ of K. If $U = U_{xy_1} \cap \cdots U_{xy_\ell}$ then

$$x \in U \quad \text{and} \quad U \cap K \subseteq (U_{xy_1} \cap \cdots \cap U_{xy_\ell}) \cap (V_{xy_1} \cup \cdots \cup V_{xy_\ell}) = \emptyset.$$

So $x \in U$ and $U \subseteq K^c$, and thus x is an interior point of K^c. So K^c is open and K is closed.

22. (compact subsets of topological spaces are not necessarily closed) Let X be a set with more than one point with topology $\mathcal{T} = \{\emptyset, X\}$. Show that every subset $A \subseteq X$ is compact but the only closed subsets of X are $\emptyset$ and X. Note that X is not Hausdorff.

23. (boundedness and completeness are not topological properties) Show that $(0, 1)$ is homeomorphic to $\mathbb{R}$.



Show that

$$\begin{array}{ll} (0,1) \text{ is bounded,} & (0,1) \text{ is not complete,} \\ \mathbb{R} \text{ is not bounded,} & \mathbb{R} \text{ is complete.} \end{array}$$

Conclude that boundedness and completeness are not topological properties.

25.2 Separability and compactness for metric spaces

1. (cover compact metric spaces have a countable base) [BR, Ch. 2 Ex. 25] Assume X is cover compact. If $n \in \mathbb{Z}_{>0}$ then $\mathcal{S}_{\frac{1}{n}} = \{B_{\frac{1}{n}}(x) \mid x \in X\}$ contains a finite subcover $\mathcal{B}_{\frac{1}{n}}$ of X. Show that the union of the $\mathcal{B}_{\frac{1}{n}}$ is a countable base of X.

2. (sequentially compact metric spaces have a countable dense set) [BR, Ch. 2 Ex. 24] Let $\delta \in \mathbb{R}_{>0}$ and $x_1 \in X$. For $i \in \mathbb{Z}_{>0}$ let

$$x_i \in X \quad \text{such that} \quad d(x_j, x_i) \geq \delta \text{ for } j = 1, 2, \ldots, i-1.$$

Use the fact that X is sequentially compact to show that this process must stop after a finite number of steps and conclude that X can be covered by a finite number of open balls of radius δ. Do this for $\delta \in \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ to obtain a countable collection of open balls whose centers form a countable dense subset of X.

3. (metric spaces with a countable base have countable open subcovers) (This exercise is one part of [BR, Ch. 2 Ex. 26].) Let (X, d) be a metric space with a countable base. Show that every open cover of X has a countable subcover.

25.3 The one point compactification

1. (The one point compactification) A *locally compact space* is a topological space $(X, \mathcal{T})$ such that X is Hausdorff and

if $x \in X$ then there exists a neighborhood N of x such that N is cover compact.

Let $(X, \mathcal{T})$ be a locally compact space and let ∞ be a symbol. The *one-point compactification* of X is

$$X^+ = X \cup \{\infty\}$$

with topology

$$\mathcal{U} = \mathcal{T} \cup \{X^+ - K \mid K \text{ is a cover compact subset of } X\}.$$

(a) Show that $\mathcal{U}$ is a topology on X^+ and that X^+ is cover compact.
(b) Show that $\mathbb{R}_{\geq 0}$ is locally compact and $(\mathbb{R}_{\geq 0})^+$ is homeomorphic to $[0, 1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$.
(c) Show that $\mathbb{R}$ is locally compact and that $\mathbb{R}^+$ is homeomorphic to $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.

25.4 Cauchy sequences and convergent sequences

1. (convergent sequences are Cauchy) Let (X, d) be a metric space and let $(x_1, x_2, \ldots)$ be a a sequence in X. Show that if there exists $x \in X$ with $\lim_{n\to\infty} x_n = x$ then $(x_1, x_2, \ldots)$ is a Cauchy sequence in X.

2. (convergent sequences are bounded) Let (X, d) be a metric space and let $(x_1, x_2, \ldots)$ be a a sequence in X. Show that if $(x_1, x_2, \ldots)$ converges in X then the set $\{x_1, x_2, \ldots\}$ is bounded.

3. (Cauchy sequences provide Cauchy filters) Let $(X, \mathcal{X})$ be a uniform space and let $(x_1, x_2, \ldots)$ be a sequence in X. Let $\mathcal{F}$ be the filter consisting of all subsets of X which contain all but a finite number of points of $\{x_1, x_2, \ldots\}$. Show that $\mathcal{F}$ is a Cauchy filter if and only if $(x_1, x_2, \ldots)$ is a Cauchy sequence.

4. (Convergent filters are Cauchy) Let $(X, \mathcal{X})$ be a uniform space and let $\mathcal{F}$ be a filter on X. Show that if $\mathcal{F}$ is convergent then $\mathcal{F}$ is Cauchy.

5. (Convergent sequences are Cauchy) Let $(X, \mathcal{X})$ be a uniform space and let $(x_1, x_2, \ldots)$ be a sequence in X. Show that if $(x_1, x_2, \ldots)$ is convergent then $(x_1, x_2, \ldots)$ is a Cauchy sequence.

6. (Cauchy sequences are not necessarily convergent) Let $X = \mathbb{R}_{(0,1)} = \{x \in \mathbb{R}\ 0 < x < 1\}$ with metric given by $d(x, y) = |x - y|$. Show that the sequence $(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$ is a Cauchy sequence in X that does not converge in X.

7. (Cauchy filters do not necessarily have limit points) Let $(X, \mathcal{X})$ be a uniform space and let $(x_1, x_2, \ldots)$ be a Cauchy sequence in X which does not have a limit point. Let $\mathcal{F}$ be the filter on X generated by the sets $\vec{x}_{\geq N} = \{x_m \mid m \in \mathbb{Z}_{\geq N}\}$ for $N \in \mathbb{Z}_{>0}$. Show that $\mathcal{F}$ is a Cauchy filter on X which does not have a limit point.

25.5 Favourite examples of complete spaces

1. ($\mathbb{R}$ is complete) Let $X = \mathbb{R}$ with metric given by $d(x, y) = |x - y|$. Show that $\mathbb{R}$ is a complete metric space.

2. ($\mathbb{R}^n$ is complete) Let $n \in \mathbb{Z}_{>0}$. Let $X = \mathbb{R}^n$ with metric given by $d(x, y) = \|x - y\|$ where $\|(x_1, \ldots, x_n)\| = \sqrt{x_1^2 + \cdots + x_n^2}$. Show that $\mathbb{R}^n$ is a complete metric space.

3. (The example $\iota\colon \mathbb{Q} \to \mathbb{R}$) The *standard metric* on $\mathbb{R}$ is

$$d\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}_{\geq 0} \qquad \text{given by} \quad d(x, y) = |y - x|,$$

where the *standard absolute value* $|\ |\colon \mathbb{R} \to \mathbb{R}_{\geq 0}$ is given by

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x \leq 0. \end{cases}$$

Show that, with respect to the standard metric, $\mathbb{R}$ is the completion of $\mathbb{Q}$.

4. (The example $\iota\colon \mathbb{R}[t] \to \mathbb{R}[[t]]$) The *t-adic metric* on $\mathbb{R}[[t]]$ is

$$d\colon \mathbb{R}[t] \times \mathbb{R}[[t]] \to \mathbb{R}_{\ge 0} \qquad \text{given by} \quad d(x,y) = e^{-v(y-x)},$$

where $e \in \mathbb{R}_{>1}$ and the *t-adic valuation* $v\colon \mathbb{R}[[t]] \to \mathbb{Z}_{\ge 0}$ is given by

$$v(p) = \max\{n \in \mathbb{Z}_{\ge 0} \mid p \in t^n\mathbb{R}[[t]]\}.$$

Show that, with respect to the t-adic metric, $\mathbb{R}[[t]]$ is the completion of $\mathbb{R}[t]$.

5. (The example $\iota\colon \mathbb{R}(t) \to \mathbb{R}((t))$) The *t-adic metric* on $\mathbb{R}((t))$ is

$$d\colon \mathbb{R}((t)) \times \mathbb{R}((t)) \to \mathbb{R}_{\ge 0} \qquad \text{given by} \quad d(x,y) = e^{-v(y-x)},$$

where $e \in \mathbb{R}_{>1}$ and the *t-adic valuation* $v\colon \mathbb{R}((t)) \to \mathbb{Z}_{ge0}$ is given by

$$v(f) = \max\{n \in \mathbb{Z}_{\ge 0} \mid f \in t^n\mathbb{R}[[t]]\}.$$

Show that, with respect to the t-adic metric, $\mathbb{R}((t))$ is the completion of $\mathbb{R}(t)$.

6. (The example $\iota\colon \mathbb{Q} \to \mathbb{Q}_p$) Let $p \in \mathbb{Z}_{>1}$ be prime. The *p-adic metric* on $\mathbb{Q}_p$ is

$$d\colon \mathbb{Q}_p \times \mathbb{Q}_p \to \mathbb{R}_{\ge 0} \qquad \text{given by} \quad d(x,y) = e^{-v_p(y-x)},$$

where $e \in \mathbb{R}_{>1}$ and the *p-adic valuation* $v_p\colon \mathbb{Q}_p \to \mathbb{Z}$ is given by

$$v_p(a) = \max\{n \in \mathbb{Z} \mid a \in p^n\mathbb{Z}_p\}.$$

Show that, with respect to the p-adic metric, $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$.

7. (The example $\iota\colon \mathbb{Z} \to \mathbb{Z}_p$) Let $p \in \mathbb{Z}_{>1}$ be prime. The *p-adic metric* on $\mathbb{Z}_p$ is

$$d\colon \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{R}_{\ge 0} \qquad \text{given by} \quad d(x,y) = e^{-v_p(y-x)},$$

where $e \in \mathbb{R}_{>1}$ and the *p-adic valuation* $v_p\colon \mathbb{Z}_p \to \mathbb{Z}_{\ge 0}$ is given by

$$v_p(a) = \max\{n \in \mathbb{Z}_{\ge 0} \mid a \in p^n\mathbb{Z}_p\}.$$

Show that, with respect to the p-adic metric, $\mathbb{Z}_p$ is the completion of $\mathbb{Z}$.

8. (ℓ^2 is not ball compact) Let $e_1 = (1,0,0,\ldots)$, $e_2 = (0,1,0,0,\ldots)$, $e_3 = (0,0,1,0,0,\ldots)$, $\ldots$ in ℓ^2. Show that
   (a) If $A = \{e_1, e_2, \ldots\}$ then $A \subseteq B_{\sqrt{2}+.001}(e_1)$ so that A is bounded.
   (b) If $A = \{e_1, e_2, \ldots\}$ and $\epsilon \in \mathbb{R}_{>0}$ with $\epsilon < \sqrt{2}$ then there do not exist a finite number of balls of radius ϵ which cover A. Thus A is not ball compact.
   (c) Show that $e_1, e_2, e_3, \ldots$ is a sequence in ℓ^2 with no cluster point.

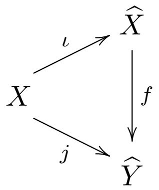
9. (ℓ^2 is Cauchy compact) Show that ℓ^2 is Cauchy compact.

25.6 Existence and uniqueness of completions

1. (isometries are injective) Show that if $\varphi\colon X \to Y$ is an isometry then φ is injective.

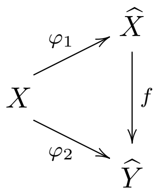
2. (isometries are not necessarily surjective) Show that if $\varphi\colon \mathbb{Q} \to \mathbb{R}$ given by $\varphi(x) = x$ is an isometry that is not surjective.

3. (uniqueness of the completion of a uniform space) Let $(X, \mathcal{X})$ be a uniform space. Show that if $(\widehat{X}, \widehat{\mathcal{X}}, \iota\colon X \to \widehat{X})$ and $(\widehat{Y}, \widehat{\mathcal{Y}}, j\colon X \to \widehat{Y})$ are completions of X then there exists a bijective uniformly continuous function $f\colon \widehat{X} \to \widehat{Y}$ such that the inverse function $f^{-1}\colon Y \to X$ is uniformly continuous and $j = f \circ \iota$.



4. (uniqueness of the completion of a metric space) Let (X, d) be a metric space. Show that if $(\widehat{X}_1, \hat{d}_1)$ with $\varphi_1\colon X \to \widehat{X}_1$ and $(\widehat{X}_2, \hat{d}_2)$ with $\varphi_2\colon X \to \widehat{X}_2$ are completions of (X, d) then there exists

$$f\colon \widehat{X}_1 \to \widehat{X}_2 \qquad \text{such that} \qquad f \text{ is an isometry, } f \text{ is a bijection, and } f \circ \varphi_1 = \varphi_2.$$



5. (existence of the completion of a metric space) Let (X, d) be a metric space. Let $(\widehat{X}, \hat{d}, \iota)$ be the metric space

$$\widehat{X} = \{\text{Cauchy sequences } \vec{x} \text{ in } X\} \qquad \text{with the function} \qquad \begin{array}{rccc} \iota\colon & X & \longrightarrow & \widehat{X} \\ & x & \longmapsto & (x, x, x, \ldots) \end{array}$$

where $\widehat{X}$ has the metric

$$d\colon \widehat{X} \times \widehat{X} \to \mathbb{R}_{\geq 0} \qquad \text{defined by} \qquad d(\vec{x}, \vec{y}) = \lim_{n\to\infty} d(x_n, y_n),$$

and *Cauchy sequences* $\vec{x} = (x_1, x_2, \ldots)$ *and* $\vec{y} = (y_1, y_2, \ldots)$ *are equal in* $\widehat{X}$,

$$\vec{x} = \vec{y} \qquad \text{if } \lim_{n\to\infty} d(x_n, y_n) = 0.$$

Show that $(\widehat{X}, \hat{d})$ with an isometry $\iota\colon X \to \widehat{X}$ such that

$$(\widehat{X}, \hat{d}) \text{ is a complete complete metric space} \qquad \text{and} \qquad \overline{\varphi(X)} = \widehat{X},$$

where $\overline{\varphi(X)}$ is the closure of the image of φ.

6. (another construction of the completion of a metric space) Let (X, d) be a metric space. The space of *bounded functions on* X is

$$B(X) = \{f : X \to \mathbb{R} \mid f(X) \text{ is bounded}\}$$

with metric $d_\infty : B(X) \times B(X) \to \mathbb{R}_{\geq 0}$ given by

$$d_\infty(f, g) = \sup\{|f(x) - g(x)| \mid x \in X\}.$$

Fix $a \in X$. Let $(\widehat{X}, \hat{d}, \iota)$ be the metric space

$$\widehat{X} = \overline{\iota(X)} \qquad \text{where} \qquad \begin{array}{cccc} \iota : & X & \to & B(X) \\ & x & \mapsto & f_x \end{array}$$

with

$$f_x : X \to \mathbb{R} \qquad \text{given by} \qquad f_x(y) = d(y, x) - d(y, a).$$

Show that ι is well defined and $(\widehat{X}, d_\infty, \iota)$ is a completion of X.

7. (existence of the completion of a uniform space) Let $(X, \mathcal{X})$ be a uniform space. A *minimal Cauchy filter on* X is a Cauchy filter on X which is minimal with respect to inclusion of filters. The *completion* of X is the uniform space

$$\widehat{X} = \{\text{minimal Cauchy filters } \hat{x} \text{ on } X\} \qquad \text{with the function} \qquad \begin{array}{cccc} \iota : & X & \longrightarrow & \widehat{X} \\ & x & \longmapsto & \mathcal{N}(x) \end{array}$$

where $\mathcal{N}(x)$ is the neighborhood filter of x, and $\widehat{X}$ has the uniformity $\widehat{\mathcal{X}}$ generated by the sets

$$\hat{V} = \{(\hat{x}, \hat{y}) \mid \text{there exists } N \in \hat{x} \cap \hat{y} \text{ such that } N \times N \subseteq V\},$$

$$\text{for } V \in \mathcal{X} \text{ such that if } (x, y) \in V \text{ then } (y, x) \in V.$$

Show that $(\widehat{X}, \widehat{\mathcal{X}})$ is a complete Hausdorff uniform space and $\iota : X \to \widehat{X}$ is a uniformly continuous function such that

> if Y is a complete Hausdorff uniform space and $f : X \to Y$ is a uniformly continuous map then there exists a unique uniformly continuous function $g : \widehat{X} \to Y$ such that $f = g \circ \iota$.

25.7 Completions and inverse limits

1. (Completions and inverse limits)

A *topological abelian group* is a topological space $(G, \mathcal{T})$ with a function

$$\begin{array}{ccc} G \times G & \longrightarrow & G \\ (g_1, g_2) & \longmapsto & g_1 + g_2 \end{array} \qquad \text{such that}$$

(a) If $g_1, g_2, g_3 \in G$ then $(g_1 + g_2) + g_3 = g_1 + (g_2 + g_3)$,

(b) There exists $0 \in G$ such that if $g \in G$ then $g + 0 = g$ and $0 + g = g$,

(c) If $g \in G$ then there exists $-g \in G$ such that $g + (-g) = 0$ and $(-g) + g = 0$,

(d) If $g_1, g_2 \in G$ then $g_1 + g_2 = g_2 + g_1$,

(e) The function
$$\begin{array}{ccc} G\times G & \longrightarrow & G \\ (g_1,g_2) & \longmapsto & g_1+g_2 \end{array} \qquad \text{is continuous, and}$$

(f) The function
$$\begin{array}{ccc} G & \longrightarrow & G \\ g & \longmapsto & -g \end{array} \qquad \text{is continuous.}$$

Assume that $\mathcal{N}(0)$, the neighborhood filter of 0 in G is countably generated (i.e. there exist $U_1, U_2, \ldots \in \mathcal{N}(0)$ such that if $P \in \mathcal{N}(0)$ then there exists $j \in \mathbb{Z}_{>0}$ such that $P \supseteq U_j$).

A *Cauchy sequence in* G is a sequence $x_1, x_2, \ldots \in G$ such that

$$\begin{array}{l} \text{if } P \in \mathcal{N}(0) \text{ then there exists } N \in \mathbb{Z}_{>0} \text{ such that} \\ \quad \text{if } r,s \in \mathbb{Z}_{\geq N} \text{ then } x_r - x_s \in P. \end{array}$$

Two Cauchy sequences $(x_1, x_2, \ldots)$ and $(y_1, y_2, \ldots)$ are *equivalent*,

$$(x_1, x_2, \ldots) \sim (y_1, y_2, \ldots), \qquad \text{if} \qquad \lim_{n\to\infty}(x_n - y_n) = 0.$$

The *completion* of G is the set of equivalence classes of Cauchy sequences in G,

$$\widehat{G} = \{\text{Cauchy sequences } (x_1, x_2, \ldots) \text{ in } G\}/\sim$$

with the function
$$\begin{array}{rccc} \varphi\colon & G & \longrightarrow & \hat{G} \\ & x & \longmapsto & (x,x,\ldots). \end{array}$$

Now assume that $G_1 \supseteq G_2 \supseteq$ are subgroups which generate $\mathcal{N}(0)$ (i.e. $G_1, G_2, \ldots \in \mathcal{N}(0)$ and if $P \in \mathcal{N}(0)$ then there exists $j \in \mathbb{Z}_{>0}$ such that $P \supseteq G_j$). A *coherent sequence* is a sequence $(\bar{x}_1, \bar{x}_2, \ldots)$ with

$$\bar{x}_n \in G/G_n \quad \text{and} \quad \pi_n(\bar{x}_{n+1}) = \bar{x}_n, \qquad \text{where} \quad \begin{array}{rcc} \pi_n\colon G/G_{n+1} & \longrightarrow & G/G_n \\ \bar{g} & \longmapsto & \bar{g}+G_n. \end{array}$$

The *inverse limit*
$$\varprojlim G/G_n \quad \text{is the set of coherent sequences.}$$

Show that the function

$$\begin{array}{rccc} \Phi\colon & \hat{G} & \longrightarrow & \varprojlim G/G_n \\ & (x_1,x_2,\ldots) & \longmapsto & (x_1+G_1, x_2+G_2, \ldots) \end{array} \qquad \text{is an isomorphism.}$$

25.7.1 Products and function spaces

1. (products of complete metric spaces are complete) Let (X, d_X) and (Y, d_Y) be complete metric space spaces. Show that $X \times Y$ with metric given by

$$d((x_1,y_1),(x_2,y_2)) = d_X(x_1,x_2) + d_Y(y_1,y_2)$$

is a complete metric space.

2. (products of complete uniform spaces are complete) Let $(X, \mathcal{X})$ and $(X, \mathcal{Y})$ be complete uniform spaces. Show that $X \times Y$ with the product uniformity is a complete uniform space. (See [Bou Ch II §3 no. 5 Proposition 10]).

3. (function spaces with complete targets are complete) Let (X, d) and (Y, ρ) be metric spaces and let
$$C_b(X, Y) = \{f \colon X \to Y \mid f \text{ is continuous and } f(X) \text{ is bounded}\}$$
with metric $d_\infty \colon C_b(X, Y) \times C_b(X, Y) \to \mathbb{R}_{\geq 0}$ given by
$$d_\infty(f, g) = \sup\{\rho(f(x), g(x)) \mid x \in X\}.$$
Show that if (Y, ρ) is a complete metric space then $C_b(X, Y)$ is a complete metric space.

4. (the metric space of bounded continuous real valued functions is complete) Let (X, d) and (Y, ρ) be metric spaces and
$$C_b(X) = \{f \colon X \to \mathbb{R} \mid f \text{ is continuous and } f(X) \text{ is bounded}\}$$
with metric $d_\infty \colon C_b(X) \times C_b(X) \to \mathbb{R}_{\geq 0}$ given by
$$d_\infty(f, g) = \sup\{|f(x) - g(x)| \mid x \in X\}.$$
Show that $C_b(X)$ is a complete metric space.

5. (If W is complete then $B(V, W)$ is complete) Let V and W be normed vector spaces and let $B(V, W)$ be the vector space of bounded linear operators from V to W with norm given by
$$\|T\| = \sup\left\{\frac{\|Tv\|}{\|v\|} \mid v \in V\right\}, \qquad \text{for } T \in B(V, W).$$
Show that if W is complete then $B(V, W)$ is complete.

6. (If Y is complete then bounded continuous functions from X to Y is complete) Let (X, d_X) and (Y, d_Y) be metric spaces and let
$$\mathcal{BC}(X, Y) = \{f \colon X \to Y \mid f \text{ is continuous and } f(X) \text{ is bounded in } Y\},$$
with $d_\infty \colon \mathcal{BC}(X, Y) \times \mathcal{BC}(X, Y) \to \mathbb{R}_{\geq 0}$ given by
$$d_\infty(f, g) = \sup\{d_Y(f(x), g(x)) \mid x \in X\}.$$
   (a) Show that $\mathcal{BC}(X, Y)$ is a metric space.
   (b) Show that if Y is a complete metric space then $\mathcal{BC}(X, Y)$ is a complete metric space.

7. (bounded real valued functions is a complete metric space) Let (X, d) be a metric space and let
$$B(X) = \{f \colon X \to \mathbb{R} \mid f(X) \text{ is bounded}\},$$
with metric $d_\infty \colon B(X) \times B(X) \to \mathbb{R}_{\geq 0}$ given by
$$d_\infty(f, g) = \sup\{|f(x) - g(x)| \mid x \in X\}.$$
Show that $B(X)$ is a complete metric space.

8. (duals of normed vector spaces are complete) Let V with $\| \ \| \colon V \to \mathbb{R}_{\geq 0}$ be a normed vector space. Show that V^*, the dual of V, is complete.

25.7.2 Banach fixed point theorem and Picard iteration

1. (Banach fixed point theorem) Let (X, d) be a metric space.
A *contraction mapping* is a function $f\colon X \to X$ such that there exists $\alpha \in \mathbb{R}_{>0}$ such that $\alpha < 1$ and
$$\text{if } x, y \in X \quad \text{then} \quad d(f(x), f(y)) \le \alpha\, d(x, y).$$
A *fixed point* of $f\colon X \to X$ is an element $x \in X$ such that $f(x) = x$.

   Let (X, d) be a complete metric space and let $f\colon X \to X$ be a contraction mapping. Let $x \in X$ and let $x_1, x_2, \ldots$ be the sequence
$$x_1 = f(x), \quad x_2 = f(f(x)), \quad x_3 = f(f(f(x))), \quad \ldots.$$
Show that the sequence $x_1, x_2, \ldots$ converges and $p = \lim_{n\to\infty} x_n$ is the unique fixed point of f.

2. (Picard iteration) Picard iteration is a method for solving equations of the the form $f(x) = x$. The process is to let
$$a_1 = \text{ your choice}, \quad a_2 = f(a_1), \quad a_3 = f(a_2), \quad \ldots.$$
If the sequence $(a_1, a_2, \ldots)$ converges and $a = \lim\limits_{n\to\infty} a_n$ then $f(a) = a$ (because $f(a_n) = a_{n+1}$ is very close to a_n for large n). To apply this technique to find a solution of $x^3 - x - 1 = 0$ proceed as follows.

   (a) Transform the equation $x^3 - x - 1 = 0$ to the form $x = f(x)$, where $f(x) = \frac{1}{x^2+1}$.
   (b) Let $a_1 = \frac{1}{2}$. Show that $a_2 = \frac{4}{5} = 0.8$.
   (c) Show that $a_3 = \frac{25}{41} \approx 0.609760976097....$
   (d) Show that $a_4 = \frac{1681}{2306} \approx 0.728967$.
   (e) Show that $a_5 \approx 0.6530046$.
   (f) Show that $a_6 \approx 0.7010582$.
   (g) Show that $a_7 \approx 0.6704737$.
   (h) Show that $a_8 \approx 0.68987635$.
   (i) Show that $a_9 \approx 0.67753918$.
   (j) Show that $a_{10} \approx 0.68537308$.
   (k) Show that $a_{11} \approx 0.680394233$.
   (l) Show that $a_{12} \approx 0.6835567$.
   (m) Show that $a_{13} \approx 0.68154722$.
   (n) Show that $a_{14} \approx 0.68282382$.
   (o) Show that $a_{15} \approx 0.6820126$.
   (p) Prove that, to 3 decimal places of accuracy,
   $x = .682$ is a solution of $x^3 + x - 1 = 0$.

3. (Picard iteration doesn't always converge) Picard iteration is a method for solving equations of the the form $f(x) = x$. The process is to let
$$a_1 = \text{ your choice}, \quad a_2 = f(a_1), \quad a_3 = f(a_2), \quad \ldots.$$
If the sequence $(a_1, a_2, \ldots)$ converges and $a = \lim\limits_{n\to\infty} a_n$ then $f(a) = a$ (because $f(a_n) = a_{n+1}$ is very close to a_n for large n). Another transformation of the equation $x^3 - x - 1 = 0$ to the form $x = f(x)$, has $f(x) = 1 - x^3$.

(a) Let $a_1 = \frac{1}{2}$. Show that $a_2 = \frac{7}{8} = 0.875$.
(b) Show that $a_3 \approx 0.330078$.
(c) Show that $a_4 \approx 0.964037$.
(d) Show that $a_4 \approx 0.104055$.
(e) Prove that $(a_1, a_2, a_3, \ldots)$ does not converge, but is oscillating between close to 1 and close to 0.

4. Which of the following maps are contractions?

(a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{-x}$;
(b) $f : [0, \infty) \to [0, \infty)$, $f(x) = e^{-x}$;
(c) $f : [0, \infty) \to [0, \infty)$, $f(x) = e^{-e^x}$;
(d) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \cos x$;
(e) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \cos(\cos x)$.

5. Let X be a complete metric space and let $f\colon X \to X$ be a contraction. Show that f has a unique fixed point.

6. Let $\alpha \in \mathbb{R}$ with $0 < \alpha < 1$. Let X be a complete metric space and let $f\colon X \to X$ be a α-contraction. Let $x \in X$, $x_0 = x$ and $x_{n+1} = f(x_n)$, for $n \in \mathbb{Z}_{\geq 0}$.

(a) Show that the sequence $x_0, x_1, x_2, \ldots$ converges in X.

Let $p = \lim_{n\to\infty} x_n$.

(b) Show that $d(x, p) \leq \dfrac{d(x, f(x))}{1-\alpha}$.
(c) Show that $f(p) = p$.

7. Let U be an open subset of $\mathbb{R}^2$. Let $f\colon U \to \mathbb{R}$ be a continuous function which satisfies the Lipschitz condition with respect to the second variable: There exists $\alpha \in \mathbb{R}_{>0}$ such that

$$\text{if } (x, y_1), (x, y_2) \in U \quad \text{then} \quad |f(x, y_1) - f(x, y_2)| \leq \alpha |y_1 - y_2|.$$

Show that if $(x_0, y_0) \in U$ then there exists $\delta \in \mathbb{R}_{>0}$ such that $y'(x) = f(x, y(x))$ has a unique solution $y\colon [x_0 - \delta, x_0 + \delta] \to \mathbb{R}$ such that $y(x_0) = y_0$.

8. Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x, y) = \frac{1}{10}(8x + 8y, x + y),\ (x, y) \in \mathbb{R}^2.$$

Recall metrics $d_1((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$, $d_2((x_1, y_1), (x_2, y_2)) = [|x_1 - x_2|^2 + |y_1 - y_2|^2]^{1/2}$ and $d_\infty((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$. Is f a contraction with respect to d_1? d_2? d_∞?

9. (a) Consider $X = (0, a]$ with the usual metric and $f(x) = x^2$ for $x \in X$. Find values of a for which f is a contraction and show that $f : X \to X$ does not have a fixed point.

(b) Consider $X = [1, \infty)$ with the usual metric and let $f(x) = x + \frac{1}{x}$ for $x \in X$. Show that $f : X \to X$ and $d(f(x), f(y)) < d(x, y)$ for all $x \neq y$, but f does not have a fixed point. Reconcile (a) and (b) with Banach fixed point theorem.

10. Let (X, d) be a complete metric space and $f : X \to X$ be a function such that

$$d(f(x), f(y)) \leq \alpha d(x, y)$$

for all $x, y \in \overline{B}(x_0, r_0)$, where $0 < \alpha < 1$ and $d(x_0, f(x_0)) \leq (1 - \alpha) \cdot r_0$. Prove that f has a unique fixed point $p \in \overline{B}(x_0, r_0)$.

11. (a) Show that there is exactly one continuous function $f : [0, 1] \to \mathbb{R}$ which satisfies the equation

$$[f(x)]^3 - e^x[f(x)]^2 + \frac{f(x)}{2} = e^x.$$

(Hint: rewrite the equation as $f(x) = e^x + \frac{1}{2}\frac{f(x)}{1 + f(x)^2}$.)

(b) Consider $C[0, a]$ with $a < 1$ and $T : C[0, a] \to C[0, a]$ given by

$$(Tf)(t) = \sin t + \int_0^t f(s)ds, \ t \in [0, a].$$

Show that T is a contraction. What is the fixed point of T?

(c) Find all $f \in C[0, \pi]$ which satisfy the equation

$$3f(t) = \int_0^t \sin(t - s)f(s)ds.$$

(d) Let $g \in C[0, 1]$. Show that there exists exactly one $f \in C[0, 1]$ which solves the equation

$$f(x) + \int_0^1 e^{x-y-1}f(y)dy = g(x), \quad \text{for all } x \in [0, 1].$$

(Hint: Consider the metric $d(f, h) = \sup\{e^{-x}|f(x) - h(x)| \mid x \in [0, 1]\}$.)

12. Call a map $f : X \to X$ a **weak contraction** if $d(f(x), f(y)) < d(x, y)$ for all $x \neq y$. Prove that if X is compact and f is a weak contraction, then f has a unique fixed point.

13. Let $a > 0$, and let

$$f(x) = \frac{1}{2}\left(x + \frac{a}{x}\right) \quad \text{for } x > 0.$$

(a) Show that $f(x) \geq \sqrt{a}$ for all $x > 0$. Hence f defines a function $f : X \to X$ where $X = [\sqrt{a}, \infty)$.

(b) Show that f is a contraction mapping when X is given the usual metric.

(c) Fix $x_0 > \sqrt{a}$ and $x_{n+1} = f(x_n)$ for all $n \geq 0$. Show that the sequence $\{x_n\}$ converges and find its limit with respect to the usual metric on $\mathbb{R}$.

14. (a) State the Banach fixed point theorem.
A mapping $f : \mathbb{R} \to \mathbb{R}$ is defined as a *contraction* if there exists a constant c with $0 \le c < 1$ such that $|f(x) - f(y)| \le c|x - y|$ for all $x, y \in \mathbb{R}$.
(b) (1) Use (a) to show that the equation $x + f(x) = a$ has a unique solution for each $a \in \mathbb{R}$.
(2) Deduce that $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = x + f(x)$ is a bijection. (This should be easy).
(3) Show that F is continuous.
(4) Show that F^{-1} is continuous. (Hence F is a homeomorphism.)

15. (a) State the Banach fixed point theorem.
(b) Let X be the interval $(0, 1/3)$ with usual Euclidean metric. Show that $f : X \to X$ defined by $f(x) = x^2$ is a contraction, but f does not have a fixed point in X. Why does this not contradict the Banach fixed point theorem?
(c) Let (X, d) be a complete metric space and $f : X \to X$. Define $g(x) = f(f(x))$, that is, $g = f \circ f$. Assume that the map $g : X \to X$ is a contraction. Prove that f has a unique fixed point.

16. Let $f : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ be given by
$$f(x) = \frac{2}{2 + x}.$$
(a) Show that f defines a contraction mapping $f : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$.
(b) Fix $x_0 \ge 0$ and $x_{n+1} = f(x_n)$ for all $n \ge 0$. Show that the sequence $\{x_n\}$ converges and find its limit with respect to the usual metric on $\mathbb{R}$.

17. Let $a \in \mathbb{R}_{>0}$. Let
$$f(x) = \frac{1}{2}\left(x + \frac{a}{x}\right), \qquad \text{for } x \in \mathbb{R}_{>0}.$$
(a) Show that if $x \in \mathbb{R}_{>0}$ then $f(x) \ge \sqrt{a}$. Hence f defines a function $f : X \to X$ where $X = [\sqrt{a}, \infty)$.
(b) Show that f is a contraction mapping when X is given the usual metric.
(c) Fix $x_0 > \sqrt{a}$ and $x_{n+1} = f(x_n)$, for $n \in \mathbb{Z}_{\ge 0}$. Show that the sequence $\{x_n\}$ converges and find its limit with respect to the usual metric on $\mathbb{R}$.

18. Consider the map $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ given by
$$f(x, y) = \tfrac{1}{10}(8x + 8y, x + y).$$
Recall metrics
$$\begin{aligned} d_1((x_1, y_1), (x_2, y_2)) &= |x_1 - x_2| + |y_1 - y_2|, \\ d_2((x_1, y_1), (x_2, y_2)) &= (|x_1 - x_2|^2 + |y_1 - y_2|^2)^{1/2}, \\ d_\infty((x_1, y_1), (x_2, y_2)) &= \max\{|x_1 - x_2|, |y_1 - y_2|\} \end{aligned}$$
If f a contraction with respect to d_1? d_2? d_∞? Prove that your answers are correct.

19. (a) State the Banach fixed point theorem.
    (b) Let $X = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 1\}$.
    Verify that the mapping $f\colon X \to X$ given by
$$f(x,y) = \big(\tfrac{1}{4}(x+y+1), \tfrac{1}{4}(x-y+1)\big)$$
    satisfies the conditions of the Banach fixed point theorem.
    (c) Find directly the unique fixed point for f.

25.8 The space L^1

A *rectangle in* $\mathbb{R}^k$ is $I_1 \times \ldots \times I_k$, where $I_1, \ldots, I_k$ are intervals in $\mathbb{R}$ and
$$\mathrm{vol}(I_1 \times \cdots \times I_k) = \mathrm{length}(I_1) \cdots \mathrm{length}(I_k).$$
A *step function* is a function $f\colon \mathbb{R}^k \to \mathbb{R}$ such that there exist $k \in \mathbb{Z}_{>0}$, $c \in \mathbb{R}$ and intervals $I_1, \ldots, I_k \subseteq \mathbb{R}$ such that
$$f(x) = \begin{cases} c, & \text{if } x \in I_1 \times \cdots \times I_k, \\ 0, & \text{otherwise.} \end{cases}$$
A *null set* is a subset $A \subseteq \mathbb{R}^k$ such that
$$\begin{array}{c}\text{if } \varepsilon \in \mathbb{R}_{>0} \text{ then there exists a sequence } R_1, R_2, \ldots \text{ of rectangles in } \mathbb{R}^k \\ \text{such that} \qquad A \subseteq \bigcup_{j \in \mathbb{Z}_{>0}} R_j \qquad \text{and} \qquad \sum_{j \in \mathbb{Z}_{>0}} \mathrm{vol}(R_j) < \varepsilon.\end{array}$$
A *full set* is the complement of a null set.

1. Let S be the set of linear combinations of step functions $f\colon \mathbb{R}^k \to \mathbb{R}$. Let
$$\|f\| = \int |f| \qquad \text{and} \qquad d(f,g) = \|f-g\|,$$
   for $f, g \in S$.
   (a) Show that $\|\ \|\colon S \to \mathbb{R}_{\ge 0}$ is not a norm on S.
   (b) Show that $d\colon S \times S \to \mathbb{R}_{\ge 0}$ is not a metric on S.

2. Let S be the set of linear combinations of step functions $f\colon \mathbb{R}^k \to \mathbb{R}$. Let $\sum_{i \in \mathbb{Z}_{>0}} f_i$ be a series in S which is norm absolutely convergent. Show that there exists a full set in $\mathbb{R}^k$ on which $\sum_{i \in \mathbb{Z}_{>0}} f_i$ converges.

3. Let S be the set of linear combinations of step functions $f\colon \mathbb{R}^k \to \mathbb{R}$. Let $\sum_{n \in \mathbb{Z}_{>0}} f_k$ be a series in S which is norm absolutely convergent. Show that $\sum_{n \in \mathbb{Z}_{>0}} f_n = 0$ almost everywhere if and only if the limit of the norms of the partial sums of f_n converge to 0.

4. Let L^1 be the set of functions which are equal almost everywhere to limits of norm absolutely convergent series in S, where S is the set of linear combinations of step functions $f\colon \mathbb{R}^k \to \mathbb{R}$. Define
$$\|f\| = \int f \qquad \text{and} \qquad d(f,g) = \|f-g\|, \quad \text{for } f, g \in L^1.$$
   (a) Show that $\|\ \|\colon L^1 \to \mathbb{R}_{\ge 0}$ is a norm on L^1.
   (b) Show that $d\colon L^1 \times L^1 \to \mathbb{R}_{\ge 0}$ is a metric on L^1.

25.9 Additional sample exam questions

1. Let (X,d) be a metric space and let $x_1, x_2, \ldots$ be a sequence in X. Show that if $(x_1, x_2, \ldots)$ is a Cauchy sequence then $\{x_1, x_2, \ldots\}$ is bounded.

2. Let (X,d) be a metric space and let $(x_1, x_2, \ldots)$ be a sequence in X. Show that if $(x_1, x_2, \ldots)$ converges then $(x_1, x_2, \ldots)$ is a Cauchy sequence.

3. Let (X,d) be a metric space and let $(x_1, x_2, \ldots)$ be a sequence in X. Show that if $(x_1, x_2, \ldots)$ is a Cauchy sequence and contains a convergent subsequence then $(x_1, x_2, \ldots)$ converges.

4. Give an example of a metric space (X,d) and a Cauchy sequence $(x_1, x_2, \ldots)$ in X that does not converge.

5. Give an example of a metric space (X,d) that is not complete.

6. Show that $\mathbb{R}$ with the usual metric is a complete metric space.

7. Let (X,d) be a complete metric space. Let $Y \subseteq X$ be a subspace of X. Show that if Y is closed then (Y,d) is complete.

8. Give an example of a metric space (X,d) and a subspace $Y \subseteq X$ such that (X,d) is a complete metric space and (Y,d) is not complete.

9. Let (X,d) be a metric space and let $Y \subseteq X$ be a subspace of X. Show that if (Y,d) is complete then Y is a closed subset of X.

10. Let $(X_1,d_1), \ldots, (X_\ell, d_\ell)$ be metric spaces and let $(X_1 \times \cdots \times X_\ell, d)$ be the product metric space. Show that if $(X_1,d_1), \ldots, (X_\ell, d_\ell)$ are complete then $(X_1 \times \cdots \times X_\ell, d)$ is complete.

11. Let (X,d) and (Y,d') be metric spaces and let $C_b(X,Y)$ be the set of bounded continuous functions $f\colon X \to Y$ with the metric $\rho\colon C_b(X,Y) \times C_b(X,Y) \to \mathbb{R}_{\geq 0}$ given by
$$\rho(f,g) = \sup\{d'(f(x), g(x)) \mid x \in X\}.$$
Show that if (Y,d') is complete then $(C_b(X,Y), \rho)$ is a complete metric space.

12. Show that the completion of $(0,1)$ with the usual metric is $[0,1]$ with the usual metric.

13. Let (X,d) and (Y,ρ) be metric spaces and let $f\colon X \to Y$ be an isometry. Show that f is injective.

14. Give an example of an isometry $f\colon X \to Y$ that is not surjective.

15. Let (X, d) be a metric space. Show that a completion of (X, d) exists.

16. Let (X, d) be a metric space. Show that the completion of (X, d) is unique (if it exists).

17. Let (X, d) be a metric space. Let $((X_1, d_1), \varphi_1)$ and $((X_2, d_2), \varphi_2)$ be completions of (X, d). Show that there is a surjective isometry $f\colon X_1 \to X_2$ such that $f \circ \varphi_1 = \varphi_2$.

18. Let $(X, \| \|)$ be a normed vector space. Show that $(X, \| \|)$ is complete if and only if every norm absolutely convergent series is convergent in X.

19. Let I be a closed and bounded interval in $\mathbb{R}$. Let $x_1, x_2, x_3, \ldots$ be a sequence in I. Show that there exists a subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ of $x_1, x_2, x_3, \ldots$ such that $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ converges in I.

20. Let X be a compact topological space. Let C be a closed subset of X. Show that C is compact.

21. Let X be a metric space and let E be a compact subset of X. Show that E is closed and bounded.

22. Let $C([0,1], \mathbb{R}) = \{f\colon [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$ and let $d(f,g) = \sup\{|f(x)-g(x)| \mid x \in [0,1]\}$.
    (a) Show that $d\colon C([0,1], \mathbb{R}) \times C([0,1], \mathbb{R})$ is a metric on $C([0, 1, \mathbb{R})$.
    (b) Let $A = \overline{B}_1(0) = \{f \in C([0,1], \mathbb{R})] \mid d(f, 0) \leq 1\}$. Show that A is closed and bounded.
    (c) Show that A is not compact.

23. Let $K \subseteq \mathbb{R}$. Show that K is compact if and only if K is closed and bounded.

24. Let (X, d) and (Y, d') be metric spaces and let $f\colon X \to Y$ be a continuous function. Let K be a compact subset of X. Show that $f(K)$ is compact in Y.

25. Let X be a compact metric space. Let $f\colon X \to \mathbb{R}$ be a continuous function. Show that f attains a maximum and a minimum value.

26. Let X be a compact metric space. Let $f\colon X \to Y$ be a continuous function. Show that f is uniformly continuous.

27. Let X be a set with the discrete metric. Show that X is compact if and only if X is finite.

28. Let X be a metric space and let $A \subseteq X$. Show that if A is totally bounded then A is bounded.

29. Let $X = \mathbb{R}$ with metric given by $d(x, y) = \min\{|x - y|, 1\}$.

    (a) Show that X is bounded.
    (b) Show that X is not totally bounded.

30. Let X be a metric space and let $A \subseteq X$. Show that the following are equivalent:

    (a) Every sequence in A has a convergent subsequence.
    (b) A is complete and totally bounded.
    (c) Every open cover of A has a finite subcover.

31. Let X be a topological space. Show that X is compact if and only if X satisfies if $\mathcal{C}$ is a collection of closed sets such that

$$\text{if } \ell \in \mathbb{Z}_{>0} \text{ and } C_1, \ldots, C_\ell \in \mathcal{C} \quad \text{then} \quad C_1 \cap \cdots \cap C_\ell \neq \emptyset$$

then $\displaystyle\bigcap_{C \in \mathcal{C}} C \neq \emptyset$.

32. Let X be a topological space and let $K \subseteq X$. Assume X is compact. Show that if K is closed then K is compact.

33. Let X be a topological space and let $K \subseteq X$. Assume X is Hausdorff. Show that if K is compact then K is closed.

34. Show that a compact Hausdorff space is normal.

35. Let X and Y be topological spaces and let $f: X \to Y$ be a continuous function. Let $\subseteq X$. Show that if K is compact then $f(K)$ is compact.

36. Let X and Y be topological spaces and let $f: X \to Y$ be a continuous function. Assume f is a bijection, X is compact and Y is Hausdorff. Show that the inverse function $f^{-1}: Y \to X$ is continuous.

37. Let $X = [0, 2\pi)$ and $Y = S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Let $f: [0, 2\pi) \to \S^1$ be given by $f(x) = (\cos x, \sin x)$.

    (a) Show that f is continuous.
    (b) Show that f is a bijection.
    (c) Show that $f^{-1}: S^1 \to [0, 2\pi)$ is not continuous.
    (d) Why does this not contradict the previous problem? FIX THIS.

38. Suppose that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in a metric space (X, d). Prove that the sequence of real numbers $\{d(x_n, y_n)\}$ converges.

39. Suppose that $\{x_n\}$ is a sequence in a metric space (X, d) such that $d(x_n, x_{n+1}) \le 2^{-n}$ for all $n \in \mathbb{Z}_{>0}$. Prove that $\{x_n\}$ is a Cauchy sequence.

40. Decide if the following metric spaces are complete:

    (a) $((0, \infty), d)$, where $d(x, y) = |x^2 - y^2|$ for $x, y \in (0, \infty)$.
    (b) $((-\pi/2, \pi/2), d)$, where $d(x, y) = |\tan x - \tan y|$ for $x, y \in (-\pi/2, \pi/2)$.

41. Let $X = (0, 1]$ be equipped with the usual metric $d(x, y) = |x - y|$. Show that (X, d) is not complete. Let $\tilde{d}(x, y) = \|\frac{1}{x} - \frac{1}{y}\|$ for $x, y \in X$. Show that $\tilde{d}$ is a metric on X that is equivalent to d, and that $(X, \tilde{d})$ is complete.

42. Suppose that (X, d) and $(Y, \widetilde{d})$ are metric spaces and that $f : X \to Y$ is a bijection such that both f and f^{-1} are uniformly continuous. Show that (X, d) is complete if and only if $(Y, \widetilde{d})$ is complete.

43. (Cantor's Intersection Theorem) Let (X, d) be a metric space and let $\{F_n\}$ be a "decreasing" sequence of non-empty subsets of X satisfying $F_{n+1} \subseteq F_n$ for all n.

    (a) Prove that if

    $$\text{(i) } (X, d) \text{ is complete,} \quad \text{(ii) each } F_n \text{ is closed,} \quad \text{(iii) } \mathrm{diam}(F_n) \to 0,$$

    then $\bigcap_{n \in \mathbb{Z}_{>0}} F_n$ consists of exactly one point.
    (b) Show that, if any of (i)-(iii) is omitted, then $\bigcap_{n \in \mathbb{Z}_{>0}} F_n$ may be empty.
    (c) Conversely, prove that if for every decreasing sequence $\{F_n\}$ of non-empty subsets satisfying (ii) and (iii), the intersection $\bigcap_{n \in \mathbb{Z}_{>0}} F_n$ is non-empty, then X is complete.

44. Let (X, d) be a complete metric space and let $f \colon X \to (0, \infty)$ be a continuous function. Prove that there exists a point x^* such that $f(y) \le 2f(x^*)$ for all $y \in B(x^*, \frac{1}{\sqrt{f(x^*)}})$.
    (*Hint:* Arguing by contradiction show that there exists a sequence $\{x_n\}$ with the following properties: $f(x_1) > 0$, $f(x_{n+1}) > 2f(x_n)$ for all $n \ge 1$ and $d(x_{n+1}, x_n) \le \frac{1}{\sqrt{f(x_n)}}$. Then show that $\{x_n\}$ is Cauchy.)

45. Let (X, d) be a complete metric space and let $(Y, \tilde{d})$ be a metric space. Let $\{f_n\}$ be a sequence of continuous functions from X to Y such that $\{f_n(x)\}$ converges for every $x \in X$. Prove that for every $\varepsilon > 0$ there exist $k \in \mathbb{Z}_{>0}$ and a non-empty open subset U of X such that $\tilde{d}(f_n(x), f_m(x)) < \varepsilon$ for all $x \in U$ and all $n, m \ge k$.

46. On $\mathbb{R}$ consider the metrics:
$$d_1(x,y) = |\arctan x - \arctan y|,$$
$$d_2(x,y) = \left|x^3 - y^3\right|.$$
With which of these metrics is $\mathbb{R}$ complete? If $(\mathbb{R}, d_i)$ is not complete find its completion.

47. Which of the following subsets of $\mathbb{R}$ and $\mathbb{R}^2$ are compact? ($\mathbb{R}$ and $\mathbb{R}^2$ are considered with the usual metrics).
   (a) $A = \mathbb{Q} \cap [0,1]$
   (b) $B = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$
   (c) $C = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}\}$
   (d) $D = \{(x,y) : |x| + |y| \le 1\}$
   (e) $E = \{(x,y) : x \ge 1 \text{ and } 0 \le y \le 1/x\}$

48. Prove that if $A_1, \ldots, A_k$ are compact subsets of a metric space (X,d), then $\bigcup_{i=1}^k A_i$ is compact.

49. Prove that if A_i is a compact subset of the metric space (X_i, d_i) for $i = 1, \ldots, k$, then $A_1 \times \cdots \times A_k$ is a compact subset of $X = X_1 \times \cdots X_k$ with the product metric d.

50. Let A be a non-empty compact subset of a metric space (X,d). Prove:
   (a) If $x \in X$, then there exists $a \in A$ such that $d(x,a) = d(x,A)$;
   (b) If $A \subseteq U$ and U is open, then there is $\varepsilon > 0$ such that $\{x \in X : d(x,A) < \varepsilon\} \subset U$.
   (c) If B is closed and $A \cap B = \emptyset$, then $d(A,B) > 0$.

   Hint: Recall that $(x,y) \mapsto d(x,y)$ is continuous from $X \times X \to [0,\infty)$.

51. Let $f : X \to \mathbb{R}$. Call a function f **upper semicontinuous**, abbreviated u.s.c., if for every $r \in \mathbb{R}$, $\{x \in X \mid f(x) < r\}$ is open. Similarly, f is **lower semicontinuous**, abbreviated l.s.c., if for every $r \in \mathbb{R}$, $\{x \in X \mid f(x) > r\}$ is open. Assume that X is compact. Show that every u.s.c. function assumes a maximum value and every l.s.c. function assumes a minimum value.

52. (a different construction of the completion of a metric space) An **equivalence relation** on a set X is a relation $\sim$ having the following three properties:
   (a) (Reflexivity) $x \sim x$ for every $x \in X$.
   (b) (Symmetry) If $x \sim y$, then $y \sim x$.
   (c) (Transitivity) If $x \sim y$ and $y \sim z$, then $x \sim z$.

   The **equivalence class** determined by x, and denoted by $[x]$, is defined by $[x] = \{y \in X : y \sim x\}$. We have $[x] = [y]$ if and only if $x \sim y$, and X is a disjoint union of these equivalence classes.

   Let (X,d) be a metric space and let X^* be the set of Cauchy sequences $\mathbf{x} = \{x_n\}$ in (X,d). Define a relation $\sim$ in X^* by declaring $\mathbf{x} = \{x_n\} \sim \mathbf{y} = \{y_n\}$ to mean $d(x_n, y_n) \to 0$.

(a) Show that $\sim$ is an equivalence relation.
Denote by $[\mathbf{x}]$ the equivalence class of $\mathbf{x} \in X^*$, and let $\widetilde{X}$ denote the set of these equivalence classes.

(b) Show that if $\mathbf{x} = \{x_n\}$ and $\mathbf{y} = \{y_n\} \in X^*$, then $\lim_{n\to\infty} d(x_n, y_n)$ exists. Show that if $\mathbf{x}' = \{x'_n\} \in [\mathbf{x}]$ and $\mathbf{y}' = \{y'_n\} \in [\mathbf{y}]$, then

$$\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x'_n, y'_n).$$

For $[\mathbf{x}], [\mathbf{y}] \in \widetilde{X}$, define

$$D([\mathbf{x}], [\mathbf{y}]) = \lim_{n\to\infty} d(x_n, y_n).$$

Note that the definition of D is unambiguous in view of the above equality.

(c) Show that $(\widetilde{X}, D)$ is a complete metric space.
Hint: Let $[\mathbf{x}^n]$ be Cauchy in $(\widetilde{X}, D)$. Then $\mathbf{x}^n = \{x_1^n, x_2^n, x_3^n, \ldots\}$ is Cauchy in (X, d). So for every $n \in \mathbb{Z}_{>0}$, there exists $k_n \in \mathbb{Z}_{>0}$ such that

$$d(x_m^n, x_{k_n}^n) < 1/n$$

for all $m \geq k_n$.
Set $\mathbf{x} = \{x_{k_1}^1, x_{k_2}^2, x_{k_3}^3, \ldots\}$. Then show that $\mathbf{x}$ is Cauchy in (X, d) and $D([\mathbf{x}^n], [\mathbf{x}]) \to 0$.

(d) If $x \in X$, let $\varphi(x)$ be the equivalence class of the constant sequence $\mathbf{x} = (x, x, x, \ldots)$. That is, $\varphi(x) = [\mathbf{x}] = [\{x, x, x, \ldots\}]$. Show that
$\varphi : X \to \varphi(X)$ is an isometry.

(e) Show that $\varphi(X)$ is dense in $(\widetilde{X}, D)$.
Hint: Let $[\mathbf{x}] \in \widetilde{X}$ with $\mathbf{x} = \{x_1, x_2, x_3, \ldots\}$. Denote by $\mathbf{x}^n$ the constant sequence $\{x_n, x_n, x_n, \ldots\}$ and show that $D([\mathbf{x}^n], [\mathbf{x}]) \to 0$.

53. Consider the following spaces:

(a) $\mathbb{R}$ with the metric $d_1(x, y) = \dfrac{|x - y|}{1 + |x - y|}$;
(b) $\mathbb{R}$ with the metric $d_2(x, y) = |\arctan x - \arctan y|$;
(c) $\mathbb{R}$ with the metric $d_3(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ if $x \neq y$.

Is $(\mathbb{R}, d_i)$ compact?

54. Use the Heine-Borel property to prove that if $f : X \to Y$ is a continuous mapping between metric spaces and X is compact then f is uniformly continuous.

55. A family $\{F_i\}_{i\in I}$ is said to have the **finite intersection property** if for every finite subset J of I, $\bigcap_{i\in J} F_i \neq \emptyset$. Show that X is compact if and only if for every family $\{F_i\}_{i\in I}$ of closed subsets of X having the finite intersection property, the intersection $\bigcap_{i\in I} F_i \neq \emptyset$.

56. Consider $C[0, 1]$ with the usual d_∞ metric. Let

$$A = \{f \in C[0, 1] \mid 0 = f(0) \leq f(t) \leq f(1) = 1 \text{ for all } t \in [0, 1]\}.$$

Show that there is no finite 1/2-net for A.

57. Show that if $A \subseteq X$ is totally bounded, then $\overline{A}$ is also totally bounded.

58. Show that a metric space (X, d) is totally bounded if and only if every sequence $\{x_n\} \subseteq X$ contains a Cauchy subsequence.

59. Let X be a totally bounded metric space and Y a metric space. Assume that $f: X \to Y$ is a bijection. Show that if f and f^{-1} are uniformly continuous, then Y is totally bounded.

60. (Lebesgue number lemma) Let (X, d) be a compact metric space and let $\{U_i\}_{i \in I}$ be an open covering of X. Prove that there exists $\delta > 0$ such that for every subset $A \subseteq X$ with $\mathrm{diam}(A) < \delta$ there exists $i \in I$ such that $A \subseteq U_i$. (δ is called a "Lebesgue number" for the covering.)

61. Let (X, d) be a compact metric space. Assume that $f: X \to X$ preserves distance, that is,

$$d(f(x), f(y)) = d(x, y)$$

for every $x, y \in X$. Show that f is a bijection. *Hint:* Assume that $f(X) \neq X$. So there exists $a \in X \setminus f(X)$. Since f is continuous and X is compact, $f(X)$ is compact. So $d(a, f(X)) = r > 0$. Consider a sequence $x_n = f^n(a)$.

62. Let X be the set of all real sequences with *finitely many non-zero terms* with the supremum metric: if $\mathbf{x} = (x_i)$ and $\mathbf{y} = (y_i)$ then $d(\mathbf{x}, \mathbf{y}) = \sup\{|x_i - y_i| : i \in \mathbb{Z}_{>0}\}$.
For each $n \in \mathbb{Z}_{>0}$, let $\mathbf{x}^n = (1, 1/2, 1/3, \ldots, 1/n, 0, 0, \ldots)$.

    (a) Show that $\{\mathbf{x}^n\}$ is a Cauchy sequence in X.
    (b) Show that $\{\mathbf{x}^n\}$ does not converge to a point in X.
    (c) Show that the completion of X is the space of all real sequences which converge to zero, with the supremum metric.

63. Let X be a nonempty set and let (Y, d) be a complete metric space. Let $f : X \to Y$ be an injective function and define

$$d_f(x, y) = d(f(x), f(y))$$

for $x, y \in X$.

    (a) Explain briefly why d_f is a metric on X.
    (b) Show that (X, d_f) is a complete metric space if $f(X)$ is a closed subset of Y.

64. (a) Define compactness for a metric space (X, d).
    (a) Let ℓ^∞ be the set of bounded real sequences with the supremum metric.
    (b) Consider the following metric spaces. Which of these spaces are compact? Give brief explanations.
        (1) The circle $\{(x, y) : x^2 + y^2 = 1\}$ with the metric induced from $\mathbb{R}^2$;
        (2) The open disk $\{(x, y) : x^2 + y^2 < 1\}$ with the metric induced from $\mathbb{R}^2$.
        (3) The closed unit ball in the space ℓ^∞.

65. (a) State the definitions of a *Cauchy sequence* and a *complete* metric space.
    (b) Let (X, d) and (Y, d') be metric spaces, and let $f : X \to Y$ be continuous with $f(X) = Y$. Show that if (X, d) is complete and $d(x, y) \le d'(f(x), f(y))$ for all $x, y, \in X$, then (Y, d') is complete.

66. Let X be a complete normed vector space over $\mathbb{R}$. A **sphere** in X is a set

$$S(a, r) = \{x \in X : d(x, a) = \|x - a\| = r\}, \qquad \text{for } a \in X \text{ and } r \in \mathbb{R}_{>0}.$$

   (a) Show that each sphere in X is nowhere dense.
   (b) Show that there is no sequence of spheres $\{S_n\}$ in X whose union is X.
   (c) Give a geometric interpretation of the result in (b) when $X = \mathbb{R}^2$ with the Euclidean norm.
   (d) Show that the result of (b) does not hold in every complete metric space X.

67. Let $X = \mathbb{R}$ with metric given by $d(x, y) = |x - y|$.

   (a) Let $A = X$. Show that A is Cauchy compact but not bounded.
   (b) Let $A = X$. Show that A is Cauchy compact but not cover compact.
   (c) Let $A = X$. Show that A is Cauchy compact but not ball compact.
   (d) Let $A = (0, 1) \subseteq X$. Show that A is ball compact and not closed in X.
   (e) Let $A = (0, 1) \subseteq X$. Show that A is ball compact and not cover compact.
   (f) Let $A = (0, 1) \subseteq X$. Show that A is ball compact and not Cauchy compact.
   (h) Let $A = (0, 1) \subseteq X$ and let $B = A$. Show that B is closed in A but B is not Cauchy compact.
   (g) Let $Y = \mathbb{R}$ with metric given by $\rho(x, y) = \min\{|x - y|, 1\}$ and let $A = Y$. Show that A is bounded but not ball compact.

68. A family $\{F_i\}_{i \in I}$ is said to have the **finite intersection property** if for every finite subset J of I, $\bigcap_{i \in J} F_i = \emptyset$. Show that X is compact if and only if for every family $\{F_i\}_{i \in I}$ of closed subsets of X having the finite intersection property, the intersection $\bigcap_{i \in I} F_i \neq \emptyset$.

69. Let $(0, 1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$ with the standard metric. Show that X is not complete, is totally bounded and is not cover compact.

70. Let (X, d) be a metric space and let $(a_1, a_2, \ldots)$ be a sequence in X.

   (a) Carefully define cluster point and limit point of $(a_1, a_2, \ldots)$.
   (b) Prove that if z is a limit point of $(a_1, a_2, \ldots)$ then z is a cluster point of $(a_1, a_2, \ldots)$.
   (c) Carefully define Cauchy sequence and convergent sequence.
   (d) Prove that if $(a_1, a_2, \ldots)$ converges then $(a_1, a_2, \ldots)$ is Cauchy.
   (e) Carefully define complete metric space.