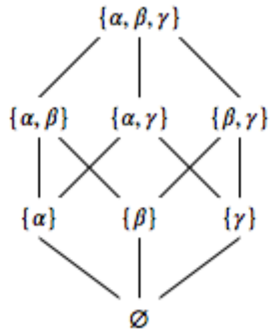
13 Orders and fields

13.1 Partially ordered sets

Let S be a set.

- A *partial order* on S is a relation $\le$ on S such that
  - (a) If $x \in A$ then $x \le x$,
  - (b) If $x, y, z \in S$ and $x \le y$ and $y \le z$ then $x \le z$, and
  - (c) If $x, y \in S$ and $x \le y$ and $y \le x$ then $x = y$.
- A *total order* on S is a partial order $\le$ such that
  - (d) If $x, y \in S$ then $x \le y$ or $y \le x$.
- A *partially ordered set*, or *poset*, is a set S with a partial order $\le$ on S.
- A *totally ordered set* is a set S with a total order $\le$ on S.



The poset of subsets of $\{\alpha, \beta, \gamma\}$ with inclusion as $\le$

Let S be a poset. Write

$$x < y \quad \text{if} \quad x \le y \text{ and } x \ne y.$$

- The *Hasse diagram* of S is the graph with vertices S and directed edges given by

$$x \to y \qquad \text{if } x \le y.$$

- A *lower order ideal of S* is a subset E of S such that

$$\text{if } y \in E \text{ and } x \in S \text{ and } x \le y \quad \text{then} \quad x \in E.$$

- The *intervals in S* are the sets

$$\begin{array}{ll}
[a,b] = \{x \in S \mid a \le x \le b\} & (a,b) = \{x \in S \mid a < x < b\} \\
[a,b) = \{x \in S \mid a \le x < b\} & (a,b] = \{x \in S \mid a < x \le b\} \\
(-\infty,b] = \{x \in S \mid x \le b\} & [a,\infty) = \{x \in S \mid a \le x\} \\
(-\infty,b) = \{x \in S \mid x < b\} & (a,\infty) = \{x \in S \mid a < x\}
\end{array}$$

  for $a, b \in S$.

13.1.1 Upper and lower bounds, sup and inf

Let S be a poset and let E be a subset of S.

- An *upper bound of* E *in* S is an element $b \in S$ such that if $y \in E$ then $y \le b$.
- A *lower bound of* E *in* S is an element $l \in S$ such that if $y \in E$ then $l \le y$.
- A *greatest lower bound of* E *in* S is an element $\inf(E) \in S$ such that
  (a) $\inf(E)$ is a lower bound of E in S, and
  (b) If $l \in S$ is a lower bound of E in S then $l \le \inf(E)$.
- A *least upper bound of* E *in* S is an element $\sup(E) \in S$ such that
  (a) $\sup(E)$ is a upper bound of E in S, and
  (b) If $b \in S$ is a upper bound of E in S then $\sup(E) \le b$.
- The set E is *bounded in* S if E has both an upper bound and a lower bound in S.

Proposition 13.1. *Let* S *be a poset and let* E *be a subset of* S*. If* $\sup(E)$ *exists then* $\sup(E)$ *is unique.*

13.2 Fields

A *field* is a set $\mathbb{F}$ with functions

$$\begin{array}{ccc} \mathbb{F}\times\mathbb{F} & \longrightarrow & \mathbb{F} \\ (a,b) & \longmapsto & a+b \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathbb{F}\times\mathbb{F} & \longrightarrow & \mathbb{F} \\ (a,b) & \longmapsto & ab \end{array}$$

such that

(Fa) If $a,b,c \in \mathbb{F}$ then $(a+b)+c = a+(b+c)$,
(Fb) If $a,b \in \mathbb{F}$ then $a+b = b+a$,
(Fc) There exists $0 \in \mathbb{F}$ such that

$$\text{if } a \in \mathbb{F} \quad \text{then} \quad 0+a = a \text{ and } a+0 = a,$$

(Fd) If $a \in \mathbb{F}$ then there exists $-a \in \mathbb{F}$ such that $a+(-a) = 0$ and $(-a)+a = 0$,
(Fe) If $a,b,c \in \mathbb{F}$ then $(ab)c = a(bc)$,
(Ff) If $a,b,c \in \mathbb{F}$ then

$$(a+b)c = ac+bc \qquad \text{and} \qquad c(a+b) = ca+cb,$$

(Fg) There exists $1 \in \mathbb{F}$ such that

$$\text{if } a \in \mathbb{F} \quad \text{then} \quad 1\cdot a = a \text{ and } a \cdot 1 = a,$$

(Fh) If $a \in \mathbb{F}$ and $a \ne 0$ then there exists $a^{-1} \in \mathbb{F}$ such that $aa^{-1} = 1$ and $a^{-1}a = 1$,
(Fi) If $a,b \in \mathbb{F}$ then $ab = ba$.

Proposition 13.2. *Let* $\mathbb{F}$ *be a field.*

(a) If $a \in \mathbb{F}$ *then* $a \cdot 0 = 0$.

(b) If $a \in \mathbb{F}$ *then* $-(-a) = a$.

(c) If $a \in \mathbb{F}$ *and* $a \ne 0$ *then* $(a^{-1})^{-1} = a$.

(d) If $a \in \mathbb{F}$ *then* $a(-1) = -a$.

(e) If $a,b \in \mathbb{F}$ *then* $(-a)b = -ab$.

(f) If $a,b \in \mathbb{F}$ *then* $(-a)(-b) = ab$.

13.2.1 Ordered fields

An *ordered field* is a field $\mathbb{F}$ with a total order $\le$ such that

(OFa) If $a, b, c \in \mathbb{F}$ and $a \le b$ then $a + c \le b + c$,
(OFb) If $a, b \in \mathbb{F}$ and $a \ge 0$ and $b \ge 0$ then $ab \ge 0$.

Proposition 13.3. *Let $\mathbb{F}$ be an ordered field.*

(a) If $a \in \mathbb{F}$ and $a > 0$ then $-a < 0$.

(b) If $a \in \mathbb{F}$ and $a \ne 0$ then $a^2 > 0$.

(c) $1 \ge 0$.

(d) If $a \in \mathbb{F}$ and $a > 0$ then $a^{-1} > 0$.

(e) If $a, b \in \mathbb{F}$ and $a \ge 0$ and $b \ge 0$ then $a + b \ge 0$.

(f) If $a, b \in \mathbb{F}$ and $0 < a < b$ then $b^{-1} < a^{-1}$.

Proposition 13.4. *Let $\mathbb{F}$ be an ordered field. Let $x, y \in \mathbb{F}$ with $x \ge 0$ and $y \ge 0$. Then*

$$x \le y \qquad \text{if and only if} \qquad x^2 \le y^2.$$

13.3 Some proofs

Proposition 13.5. *Let $\mathbb{F}$ be a field.*

(a) If $a \in \mathbb{F}$ then $a \cdot 0 = 0$.

(b) If $a \in \mathbb{F}$ then $-(-a) = a$.

(c) If $a \in \mathbb{F}$ and $a \ne 0$ then $(a^{-1})^{-1} = a$.

(d) If $a \in \mathbb{F}$ then $a(-1) = -a$.

(e) If $a, b \in \mathbb{F}$ then $(-a)b = -ab$.

(f) If $a, b \in \mathbb{F}$ then $(-a)(-b) = ab$.

Proof.

(a) Assume $a \in \mathbb{F}$.

$$\begin{aligned} a \cdot 0 &= a \cdot (0 + 0), \quad \text{by (Fc)}, \\ &= a \cdot 0 + a \cdot 0, \quad \text{by (Ff)}. \end{aligned}$$

Add $-a \cdot 0$ to each side and use (Fd) to get $0 = a \cdot 0$.

(b) Assume $a \in \mathbb{F}$.
By (Fd),

$$-(-a) + (-a) = 0 = a + (-a).$$

Add $-a$ to each side and use (Fd) to get $-(-a) = a$.

(c) Assume $a \in \mathbb{F}$ and $a \ne 0$.
By (Fh),

$$(a^{-1})^{-1} \cdot a^{-1} = 1 = a \cdot a^{-1}.$$

Multiply each side by a and use (Fh) and (Fg) to get $(a^{-1})^{-1} = a$.

(d) Assume $a \in \mathbb{F}$.
By (Ff),
$$a(-1) + a \cdot 1 = a(-1+1) = a \cdot 0 = 0,$$
where the last equality follows from part (a).
So, by (Fg), $a(-1) + a = 0$.
Add $-a$ to each side and use (Fd) and (Fc) to get $a(-1) = -a$.

(e) Assume $a, b \in \mathbb{F}$.
$$\begin{aligned}(-a)b + ab &= (-a+a)b, \quad \text{by (Ff)},\\ &= 0 \cdot b, \quad \text{by (Fd)},\\ &= 0, \quad \text{by part (a)}.\end{aligned}$$
Add $-ab$ to each side and use (Fd) and (Fc) to get $(-a)b = -ab$.

(f) Assume $a, b \in \mathbb{F}$.
$$\begin{aligned}(-a)(-b) &= -(a(-b)), \quad \text{by (e)},\\ &= -(-ab), \quad \text{by (e)},\\ &= ab, \quad \text{by part (b)}.\end{aligned}$$

□

Proposition 13.6. *Let $\mathbb{F}$ be an ordered field.*

(a) If $a \in \mathbb{F}$ and $a > 0$ then $-a < 0$.

(b) If $a \in \mathbb{F}$ and $a \neq 0$ then $a^2 > 0$.

(c) $1 \geq 0$.

(d) If $a \in \mathbb{F}$ and $a > 0$ then $a^{-1} > 0$.

(e) If $a, b \in \mathbb{F}$ and $a \geq 0$ and $b \geq 0$ then $a + b \geq 0$.

(f) If $a, b \in \mathbb{F}$ and $0 < a < b$ then $b^{-1} < a^{-1}$.

Proof.

(a) Assume $a \in \mathbb{F}$ and $a > 0$.
Then $a + (-a) > 0 + (-a)$, by (OFb).
So $0 > -a$, by (Fd) and (Fc).

(b) Assume $a \in \mathbb{F}$ and $a \neq 0$.
Case 1: $a > 0$.
Then $a \cdot a > a \cdot 0$, by (OFb).
So $a^2 > 0$, by part (a).
Case 2: $a < 0$.
Then $-a > 0$, by part (a).
Then $(-a)^2 > 0$, by Case 1.
So $a^2 > 0$, by Proposition 36.4 (f).

(c) To show: $1 \geq 0$.
$1 = 1^2 \geq 0$, by part (b).

(d) Assume $a \in \mathbb{F}$ and $a > 0$.

By part (b), $a^{-2} = (a^{-1})^2 > 0$.
So $a(a^{-1})^2 > a \cdot 0$, by (OFb).
So $a^{-1} > 0$, by (Fh) and Proposition 36.4 (a).

(e) Assume $a, b \in \mathbb{F}$ and $a \geq 0$ and $b \geq 0$.

$$\begin{aligned} a + b &\geq 0 + b, \quad \text{by (OFa)}, \\ &\geq 0 + 0, \quad \text{by (OFa)}, \\ &= 0, \quad \text{by (Fc)}. \end{aligned}$$

(f) Assume $a, b \in \mathbb{F}$ and $0 < a < b$.
So $a > 0$ and $b > 0$.
Then, by part (d), $a^{-1} > 0$ and $b^{-1} > 0$.
Thus, by (OFb), $a^{-1}b^{-1} > 0$.
Since $a < b$, then $b - a > 0$, by (OFa).
So, by (OFb), $a^{-1}b^{-1}(b - a) > 0$.
So, by (Fh), $a^{-1} - b^{-1} > 0$.
So, by (OFa), $a^{-1} > y^{-1}$.

□

Proposition 13.7. *Let $\mathbb{F}$ be an ordered field. Let $x, y \in \mathbb{F}$ with $x \geq 0$ and $y \geq 0$. Then*

$$x \leq y \qquad \text{if and only if} \qquad x^2 \leq y^2.$$

Proof. Assume $x, y \in \mathbb{F}$ and $x \geq 0$ and $y \geq 0$.

To show: (a) If $x \leq y$ then $x^2 \leq y^2$.
(b) if $x^2 \leq y^2$ then $x \leq y$.

(a) Assume $y \geq x$.
Then $y - x \geq 0$ and, since $x \geq 0$ and $y \geq 0$ then $x + y \geq 0$.
So $(x - y)(x + y) \geq 0 \cdot (x + y)$.
So $y^2 - x^2 \geq 0$.
So $y^2 \geq x^2$.

(b) Assume $x^2 \leq y^2$.
Then $y^2 + (-x)^2 \geq x^2 + (-x^2) = 0$.
So $y^2 - x^2 \geq 0$.
So $(y - x)(y + x) \geq 0$.
Since $x \geq 0$ and $y \geq 0$ then $x + y \geq 0$.
Case 1: $x + y \neq 0$.
Then $x + y > 0$ and $(x + y)^{-1} > 0$.
So $(y - x)(y + x)(y + x)^{-1} \geq 0(x + y)^{-1} = 0$.
So $(y - x) \geq 0$.
So $y \geq x$.
Case 2: $x + y = 0$.
Then $x = 0$ and $y = 0$ (since $x \geq 0$ and $y \geq 0$).
So $y \geq x$.

□

13.4 Notes and references

Fundamental definitions and properties of partially ordered sets are treated thoroughly in [Bou, Ens Ch. III]. The exposition of Stanley [St, Ch. 3] has an inspiring point of view and a wealth of information on the subject of posets. The definition of partially ordered set differs slightly depending on the author: Bourbaki replace axiom (a) in the definition by: If $x, y \in S$ and $x \leq y$ then $x \leq x$ and $y \leq y$. Bourbaki defines a preorder to be a partial order except without the axiom (b).

The set of subsets of a set S forms a partially ordered set under inclusion $\subseteq$. This is the favorite example of a partial order which is not a total order. The union $\cup$ and intersection $\cap$ make the set of subsets of S into a Boolean algebra. References for Boolean algebras are Birkhoff [Brk, Chapt X] and Stanley [St, §3.4]; in particular, the conditions for the finite Boolean algebra B_n are found in Stanley [St, p. 107-108].

The orders on the number systems $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ are indispensible for ordinary daily measurements. Perhaps surprisingly, there is no partial order on $\mathbb{C}$ which makes $\mathbb{C}$ an ordered field.

The definitions of *left filtered* and *right filtered* are used in the theory of inverse and direct limits. The definitions and examples of upper and lower bounds, suprema and infima, maxima and minima, and largest and smallest element, are a natural way to introduce students to analyses and proofs of existence and uniqueness. Directed sets are the generalization of sequences used to define nets which, in turn, provide a general method for formalizing the notion of a limit (see notes of Arun Ram on filters and nets).