11 Limits and Topologies

11.1 Spaces

The point of this section is to introduce topological spaces and metric spaces and to explain how to make a metric space into a topological space.

11.1.1 Topological spaces

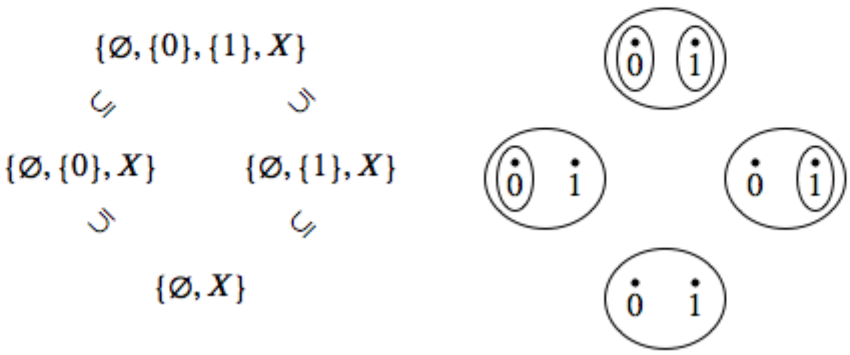
A *topological space* is a set X with a specification of the *open* subsets of X where it is required that

(a) $\emptyset$ is open in X and X is open in X,
(b) Unions of open sets in X are open in X,
(c) Finite intersections of open sets in X are open in X.

In other words, a *topology* on X is a set $\mathcal{T}$ of subsets of X such that

(a) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$,
(b) If $\mathcal{S} \subseteq \mathcal{T}$ then $\left(\bigcup_{U\in\mathcal{S}} U\right) \in \mathcal{T}$,
(c) If $\ell \in \mathbb{Z}_{>0}$ and $U_1, U_2, \ldots, U_\ell \in \mathcal{T}$ then $U_1 \cap U_2 \cap \cdots \cap U_\ell \in \mathcal{T}$.

A *topological space* $(X, \mathcal{T})$ is a set X with a topology $\mathcal{T}$ on X. An *open set in* X is a set in $\mathcal{T}$.

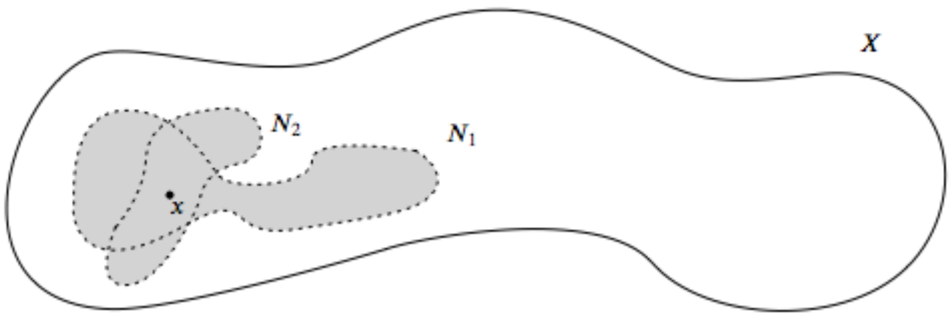


The four possible topologies on $X = \{0, 1\}$.

In a topological space, perhaps even more important than the open sets are the neighborhoods. Let $(X, \mathcal{T})$ be a topological space. Let $x \in X$. The *neighborhood filter of* x is

$$\mathcal{N}(x) = \{N \subseteq X \mid \text{there exists } U \in \mathcal{T} \text{ such that } x \in U \text{ and } U \subseteq N\}. \tag{11.1}$$

A *neighborhood of* x is a set in $\mathcal{N}(x)$.



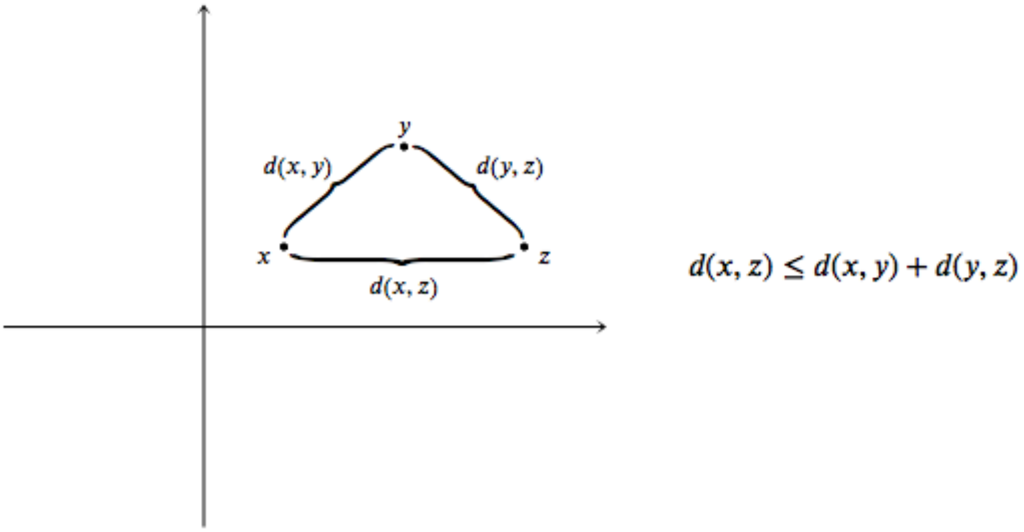
Neighborhoods of x.

11.1.2 Metric spaces

A *strict metric space* is a set X with a function $d\colon X \times X \to \mathbb{R}_{\geq 0}$ such that

(a) (diagonal condition) If $x \in X$ then $d(x, x) = 0$,
(b) (diagonal condition) If $x, y \in X$ and $d(x, y) = 0$ then $x = y$,
(c) (symmetry condition) If $x, y \in X$ then $d(x, y) = d(y, x)$,
(d) (the triangle inequality) If $x, y, z \in X$ then $d(x, y) \leq d(x, z) + d(z, y)$.

Conditions (a) and (b) are equivalent to $d^{-1}(0) = \Delta(X)$, where the *diagonal of* X is $\Delta(X) = \{(x, x) \mid x \in X\}$ and $d^{-1}(0) = \{(x, y) \in X \times X \mid d(x, y) = 0\}$.



Distances between points in the metric space $\mathbb{R}^2$.

11.1.3 Making metric spaces into topological spaces

Let $\mathbb{E} = \{10^{-k} \mid k \in \mathbb{Z}_{>0}\}$. The set $\mathbb{E}$ is the *accuracy set.* Specifying an element of $\mathbb{E}$ specifies the desired number of decimal places of accuracy.

Let (X, d) be a strict metric space. Let $x \in X$ and let $\epsilon \in \mathbb{E}$. The *open ball of radius* ϵ *at* x is

$$B_\epsilon(x) = \{y \in X \mid d(x, y) < \epsilon\}.$$

The *neighborhood filter of an element* $x \in X$ is

$$\mathcal{N}(x) = \{N \subseteq X \mid \text{there exists } \epsilon \in \mathbb{E} \text{ such that } B_\epsilon(x) \subseteq N\}.$$

The *metric space topology on* X is

$$\mathcal{T} = \{U \subseteq X \mid \text{if } x \in U \text{ then there exists } \epsilon \in \mathbb{E} \text{ such that } B_\epsilon(x) \subseteq U\}.$$

The following characterization of the metric space topology is frequently used as the definition of the metric space topology.

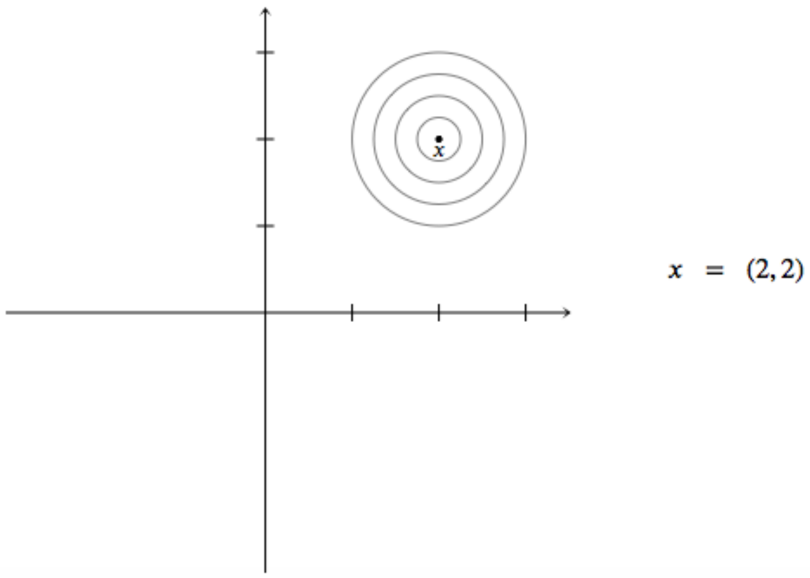
Proposition 11.1. *Let* (X, d) *be a strict metric space.*

$$\textit{Let}\quad \mathbb{E} = \{10^{-k} \mid k \in \mathbb{Z}_{>0}\} \quad \textit{and let} \quad \mathcal{B} = \{B_\epsilon(x) \mid \epsilon \in \mathbb{E} \textit{ and } x \in X\}.$$

Let $\mathcal{T}$ be the metric space topology on X. Let $U \subseteq X$. Then $U \in \mathcal{T}$ if and only if

$$\text{there exists } \mathcal{S} \subseteq \mathcal{B} \text{ such that } \quad U = \bigcup_{B \in \mathcal{S}} B.$$

Proof. (Sketch) If $U = \bigcup_{B\in\mathcal{S}} B$ and $x \in U$ then there exists $B_\delta(y) \in \mathcal{S}$ with $x \in B_\delta(y)$. Letting $\epsilon < \delta - d(x,y)$ then $B_\epsilon(x) \subseteq U$. So $U \in \mathcal{T}$. □



Generators of the neighborhood filter of $x = (2,2)$ in the metric space $\mathbb{R}^2$.

11.2 Continuous functions, interiors and closures

11.2.1 Interiors and closures

Let $(X, \mathcal{T})$ be a topological space. An *open set in* X is a subset U of X such that $U \in \mathcal{T}$. A *closed set in* X is a subset C of X such that the complement of C is an open set in X, i.e.

$$C \text{ is closed if} \qquad X - C = \{x \in X \mid x \notin C\} \quad \text{is an open set in } X.$$

Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$.
The *interior* of A is the subset A° of X such that

(a) A° is open in X and $A^\circ \subseteq A$,
(b) If U is open X and $U \subseteq A$ then $U \subseteq A^\circ$.

The *closure of* A is the subset $\overline{A}$ of X such that

(a) $\overline{A}$ is closed in X and $\overline{A} \supseteq A$,
(b) If C is closed in X and $C \supseteq A$ then $C \supseteq \overline{A}$.

Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$.
An *interior point of* A is a element $x \in X$ such that

$$\text{there exists } N \in \mathcal{N}(x) \text{ such that } N \subseteq A.$$

A *close point* to A is an element $x \in X$ such that

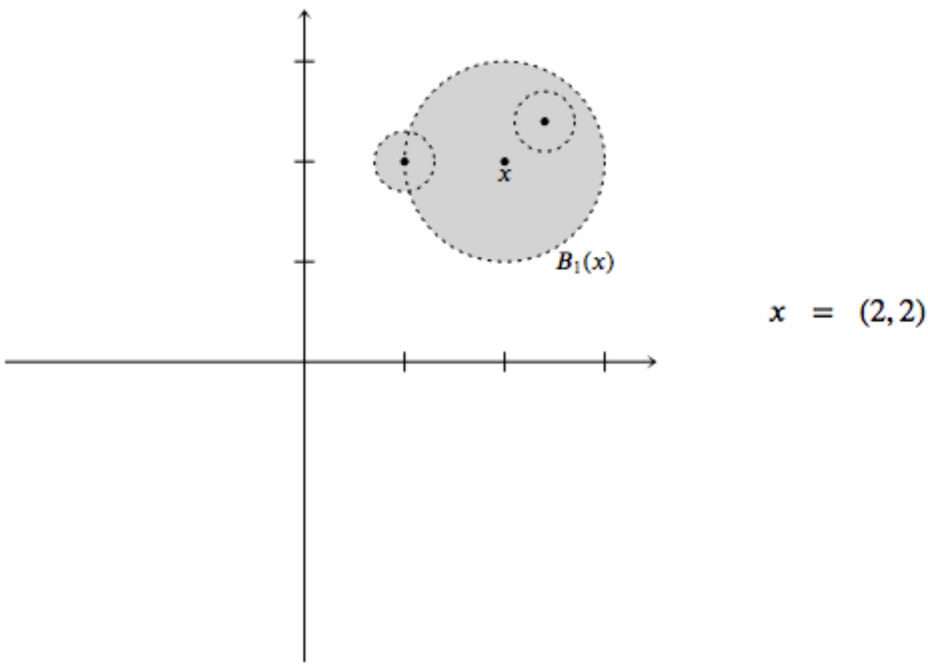
$$\text{if } N \in \mathcal{N}(x) \text{ then } N \cap A \neq \emptyset.$$

Proposition 11.2. *Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$.*

(a) The interior of A is the set of interior points of A.

(b) The closure of A is the set of close points of A.

Proof. (Sketch) For part (a): Let $I = \{\text{interior points of } A\}$ and use the definitions to show that $I \subseteq A^\circ$ and $A^\circ \subseteq I$. Part (b) is obtained from part(a) by carefully taking complements. □



An interior point and a close point of $B_1(x)$ where $x = (2, 2)$ in $\mathbb{R}^2$.

11.2.2 Continuous functions

Continuous functions are for comparing topological spaces.

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A *continuous function from X to Y* is a function $f: X \to Y$ such that

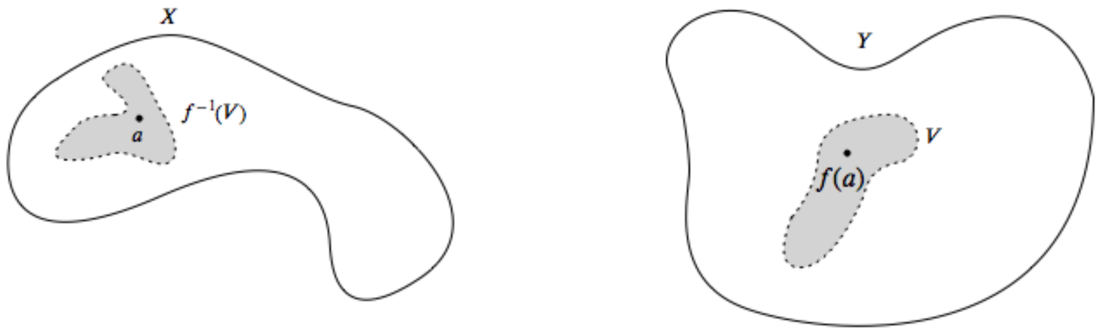
$$\text{if } V \text{ is an open set of } Y \quad \text{then} \quad f^{-1}(V) \text{ is an open set of } X,$$

where $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$. An *isomorphism of topological spaces*, or *homeomorphism*, is a continuous function $f: X \to Y$ such that the inverse function $f^{-1}: Y \to X$ exists and is continuous.

Let X and Y be topological spaces and let $a \in X$. A function $f: X \to Y$ is *continuous at a* if f satisfies the condition

$$\text{if } V \text{ is a neighborhood of } f(a) \text{ in } Y \quad \text{then} \quad f^{-1}(V) \text{ is a neighborhood of } a \text{ in } X,$$

i.e. if $V \in \mathcal{N}(f(a))$ then $f^{-1}(V) \in \mathcal{N}(a)$.



Proposition 11.3. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and let $f: X \to Y$ be a function. Then f is continuous if and only if f satisfies*

$$\textit{if } a \in X \qquad \textit{then} \quad f \textit{ is continuous at } a.$$

Proof. (Sketch) This is a combination of the definitions of continuous, continuous at a, and the definition of $\mathcal{N}(a)$ as in (11.1). □

11.3 Limits in topological spaces

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let $f: X \to Y$ be a function and let $a \in X$ and $y \in Y$. Write

$$y = \lim_{x \to a} f(x) \qquad \text{if } f \text{ satisfies:} \quad \begin{array}{l} \text{if } N \in \mathcal{N}(y) \quad \text{then} \\ \quad \text{there exists } P \in \mathcal{N}(a) \text{ such that } N \supseteq f(P). \end{array}$$

Assume $a \in X$ such that $a \in \overline{X - \{a\}}$ (in English: a is in the closure of the complement of $\{a\}$ so that a is not an isolated point). Write

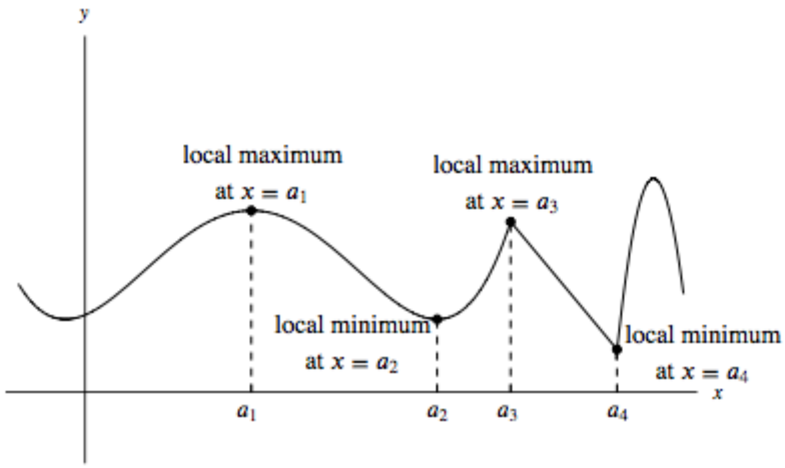
$$y = \lim_{\substack{x \to a \\ x \neq a}} f(x) \qquad \text{if } f \text{ satisfies:} \quad \begin{array}{l} \text{if } N \in \mathcal{N}(y) \quad \text{then} \\ \quad \text{there exists } P \in \mathcal{N}(a) \text{ such that } N \supseteq f(P - \{a\}). \end{array}$$

For example, using the standard topology on $\mathbb{R}$, the function $f: \mathbb{R} \to \mathbb{R}$ given by

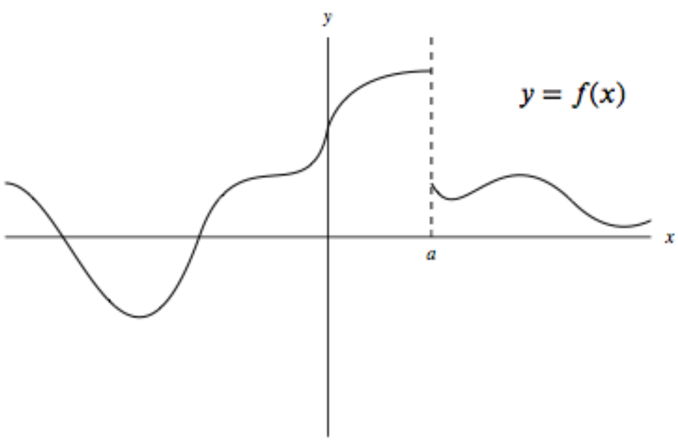
$$f(x) = \begin{cases} 2, & \text{if } x \neq 0, \\ 4, & \text{if } x = 0, \end{cases} \qquad \text{has} \qquad \lim_{\substack{x \to 0 \\ x \neq 0}} f(x) = 2 \quad \text{and} \quad \lim_{x \to 0} f(x) \text{ does not exist,}$$

and, using the subspace topology on $\{0, 1\}$ (a subspace of $\mathbb{R}$), the function $g: \{0, 1\} \to \mathbb{R}$ given by

$$g(x) = 2, \qquad \text{has} \qquad \lim_{x \to 0} f(x) = 2 \quad \text{and} \quad \lim_{\substack{x \to 0 \\ x \neq 0}} f(x) \text{ is not defined.}$$



$f\colon \mathbb{R} \to \mathbb{R}$ is continuous



$f\colon \mathbb{R} \to \mathbb{R}$ is not continuous at a

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.

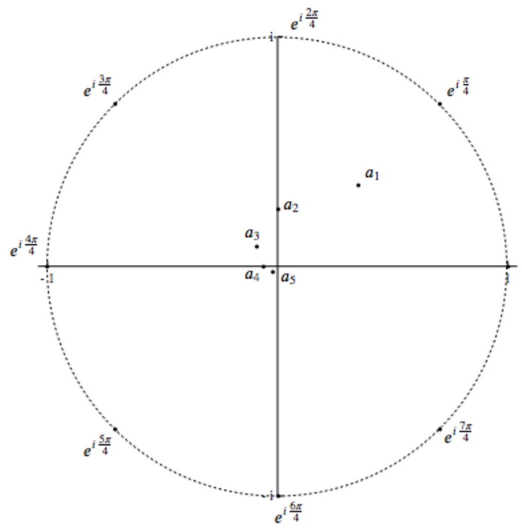
$$\text{A } \textit{sequence in } X \text{ is} \qquad \text{a function} \quad \begin{array}{rccc} \vec{x}\colon & \mathbb{Z}_{>0} & \to & X \\ & n & \mapsto & x_n \end{array}$$

Let $(X, \mathcal{T})$ be a topological space. Let $(x_1, x_2, \ldots)$ be a sequence in X and let $z \in X$. Write

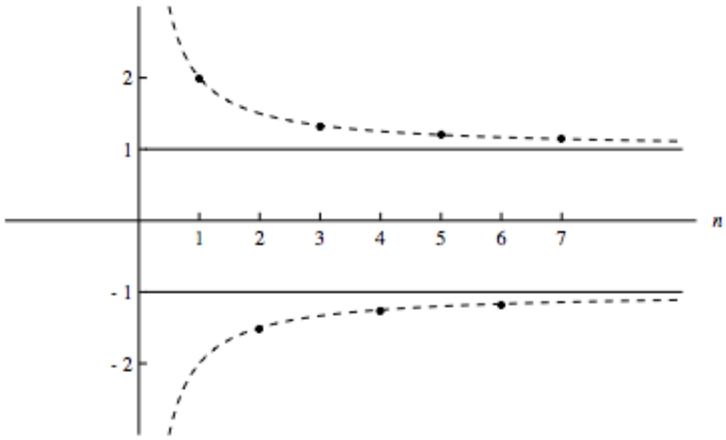
$$z = \lim_{n\to\infty} x_n \quad \text{if } (x_1, x_2, \ldots) \text{ satisfies:} \quad \begin{array}{l} \text{if } N \in \mathcal{N}(z) \quad \text{then } N \text{ contains all but} \\ \quad \text{a finite number of elements of } \{x_1, x_2, \ldots\}. \end{array}$$

More precisely,

$$z = \lim_{n\to\infty} x_n \quad \text{if } (x_1, x_2, \ldots) \text{ satisfies:} \quad \begin{array}{l} \text{if } N \in \mathcal{N}(z) \quad \text{then there exists } \ell \in \mathbb{Z}_{>0} \\ \quad \text{such that } N \supseteq \{x_\ell, x_{\ell+1}, \ldots\}. \end{array}$$



The spiral sequence $a_n = \left(\frac{1}{2}e^{i\pi/4}\right)^n$ in $\mathbb{C}$ has limit point 0



The sequence $a_n = (-1)^{n-1}(1+\frac{1}{n})$ in $\mathbb{R}$ has cluster points at 1 and at -1

11.3.1 Limits and continuity

Proposition 11.4. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let $f\colon X \to Y$ be a function.*

(a) Let $a \in X$. Then

$$f \text{ is continuous at } a \qquad \text{if and only if} \qquad \lim_{x\to a} f(x) = f(a).$$

(b) Let $a \in X$ such that $a \in \overline{X - \{a\}}$. Then

$$f \text{ is continuous at } a \qquad \text{if and only if} \qquad \lim_{\substack{x\to a \\ x\neq a}} f(x) = f(a).$$

Proof. (Sketch) The notation $\lim_{x\to a} f(x) = f(a)$ means that if $N \in \mathcal{N}(f(a))$ then $f^{-1}(N) \supseteq P$, where $P \in \mathcal{N}(a)$. But then $f^{-1}(N) \in \mathcal{N}(a)$. □

11.3.2 Limits in metric spaces

$$\text{Let } \mathbb{E} = \{10^{-1}, 10^{-2}, \ldots\}.$$

Proposition 11.5. *Let (X, d_X) and (Y, d_Y) be strict metric spaces. Let $f\colon X \to Y$ be a function and let $y \in Y$.*

(a) Let $a \in X$. Then

$$\lim_{x\to a} f(x) = y \quad \textit{if and only if} \quad f \textit{ satisfies}$$

$$\begin{gathered}\textit{if } \epsilon \in \mathbb{E} \textit{ then there exists } \delta \in \mathbb{E} \textit{ such that} \\ \textit{if } x \in X \textit{ and } d_X(x,a) < \delta \quad \textit{then} \quad d_Y(f(x), y) < \epsilon.\end{gathered}$$

(b) Let $a \in X$ be such that $a \in \overline{X - \{a\}}$. Then

$$\lim_{\substack{x\to a\\ x\neq a}} f(x) = y \quad \textit{if and only if} \quad f \textit{ satisfies}$$

$$\begin{gathered}\textit{if } \epsilon \in \mathbb{E} \textit{ then there exists } \delta \in \mathbb{E} \textit{ such that} \\ \textit{if } x \in X \textit{ and } 0 < d_X(x,a) < \delta \quad \textit{then} \quad d_Y(f(x), y) < \epsilon.\end{gathered}$$

(c) Let $(x_1, x_2, \ldots)$ be a sequence in X and let $z \in X$. Then

$$\lim_{n\to\infty} x_n = z \quad \textit{if and only if} \quad (x_1, x_2, \ldots) \textit{ satisfies}$$

$$\begin{gathered}\textit{if } \epsilon \in \mathbb{E} \textit{ then there exists } \ell \in \mathbb{Z}_{>0} \textit{ such that} \\ \textit{if } n \in \mathbb{Z}_{\geq \ell} \quad \textit{then} \quad d(x_n, z) < \epsilon.\end{gathered}$$

Proof. (Sketch) The proof is accomplished by a careful conversion of the definitions of the limits using the definition of the metric space topology and the definition of the open ball $B_\epsilon(y)$ of radius ϵ centered at y. □

11.3.3 Limits of sequences capture closure and continuity in metric spaces

Theorem 11.6. *(Closure in metric spaces) Let (X, d) be a strict metric space and let $\mathcal{T}_X$ be the metric space topology on X. Let $A \subseteq X$. Then*

$$\overline{A} = \big\{ z \in X \mid \textit{there exists a sequence } (a_1, a_2, \ldots) \textit{ in } A \textit{ such that } z = \lim_{n\to\infty} a_n \big\},$$

where $\overline{A}$ is the closure of A in X.

Proof. (Sketch) If z is a close point to A then a sequence $(a_1, a_2, \ldots)$ such that

$$a_1 \in B_{0.1}(z) \cap A, \quad a_2 \in B_{0.01}(z) \cap A, \quad a_3 \in B_{0.001}(z) \cap A, \quad \ldots,$$

will have $z = \lim_{n\to\infty} a_n$. □

Theorem 11.7. *(Continuity for metric spaces) Let (X, d_X) and (Y, d_Y) be strict metric spaces. Let $\mathcal{T}_X$ be the metric space topology on X and let $\mathcal{T}_Y$ be the metric space topology on Y. Let $f\colon X \to Y$ be a function. Then f is continuous if and only if f satisfies*

$$\textit{if } (x_1, x_2, \ldots) \textit{ is a sequence in } X \textit{ and } \lim_{n\to\infty} x_n \textit{ exists} \quad \textit{then} \quad f\Big(\lim_{n\to\infty} x_n\Big) = \lim_{n\to\infty} f(x_n).$$

Proof. (Sketch) The $\Rightarrow$ implication is similar to the proof of Theorem 11.4. For the $\Leftarrow$ implication prove the contrapositive: If f is not continuous at a then there exists $N \in \mathcal{N}(f(a))$ such that $f^{-1}(N) \notin \mathcal{N}(a)$ and letting

$$x_1 \in B_{0.1}(a) \cap f^{-1}(N)^c, \quad x_2 \in B_{0.01}(a) \cap f^{-1}(N)^c, \quad \ldots$$

produces a sequence such that $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} f(x_n) \neq f(a)$. □

11.4 Limits of sequences capture closure and continuity in topological spaces with countably generated neighborhood filters

A topological space $(X, \mathcal{T})$ has *countably generated neighborhood filters*, or is *first countable*, if $(X, \mathcal{T})$ satisfies:

$$\begin{array}{l}\text{if } x \in X \text{ then there exist subsets } B_1, B_2, \ldots \text{ of } X \text{ such that} \\ \qquad \mathcal{N}(x) = \{N \subseteq X \mid \text{there exists } k \in \mathbb{Z}_{>0} \text{ such that } N \supseteq B_k\}.\end{array}$$

Theorem 11.8. *(Closure in topological spaces with countably generated neighborhood filters) Let $(X, \mathcal{T})$ be a topological space with countably generated neighborhood filters. Let $A \subseteq X$. Then*

$$\overline{A} = \big\{z \in X \mid \textit{there exists a sequence } (a_1, a_2, \ldots) \textit{ in } A \textit{ such that } z = \lim_{n\to\infty} a_n\big\},$$

Proof. (Sketch) If z is a close point to A and $B_1, B_2, \ldots$ are generators of $\mathcal{N}(z)$ then a sequence $(a_1, a_2, \ldots)$ such that

$$a_1 \in B_1(z) \cap A, \quad a_2 \in B_2(z) \cap A, \quad a_3 \in B_3(z) \cap A, \quad \ldots,$$

will have $z = \lim_{n\to\infty} a_n$. □

Theorem 11.9. *(Continuity for topological spaces with countably generated neighborhood filters) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and assume that $(X, \mathcal{T}_X)$ has countably generated neighborhood filters. Let $f\colon X \to Y$ be a function. Then f is continuous if and only if f satisfies*

$$\textit{if } (x_1, x_2, \ldots) \textit{ is a sequence in } X \textit{ and } \lim_{n\to\infty} x_n \textit{ exists} \quad \textit{then} \quad f\Big(\lim_{n\to\infty} x_n\Big) = \lim_{n\to\infty} f(x_n).$$

Proof. (Sketch) The proof is similar to the proof of Theorem 11.7 except with generators $B_1, B_2, \ldots$ of $\mathcal{N}(a)$ replacing the open balls $B_{0.1}(a), B_{0.01}(a), \ldots$. □

11.5 Some proofs

11.5.1 Alternative characterization of the metric space topology

Proposition 11.10. *Let (X, d) be a strict metric space. Let*

$$\mathbb{E} = \{10^{-1}, 10^{-2}, \ldots\} \quad \textit{and let} \quad \mathcal{B} = \{B_\epsilon(x) \mid \epsilon \in \mathbb{E} \textit{ and } x \in X\},$$

the set of open balls in X. Let $\mathcal{T}$ be the metric space topology on X. Let $U \subseteq X$. Then $U \in \mathcal{T}$ if and only if

$$\textit{there exists } \mathcal{S} \subseteq \mathcal{B} \textit{ such that} \quad U = \bigcup_{B \in \mathcal{S}} B.$$

Proof.

$\Leftarrow$: Assume $U = \bigcup_{B\in\mathcal{S}} B$.
To show: $U \in \mathcal{T}$.
To show: If $x \in U$ then there exists $\epsilon \in \mathbb{E}$ such that $B_\epsilon(x) \subseteq U$.
Assume $x \in U$.
Since $U = \bigcup_{B\in\mathcal{S}} B$ then there exists $B \in \mathcal{S}$ such that $x \in B$.
By definition of $\mathcal{B}$ there exists $\delta \in \mathbb{E}$ and $y \in X$ such that $B = B_\delta(y)$.
Since $x \in B = B_\delta(y)$ then $d(x, y) < \delta$.
Let $\epsilon = 10^{-k}$, where $k \in \mathbb{Z}_{>0}$ is such that $0 < 10^{-k} < \delta - d(x, y)$.
To show: $B_\epsilon(x) \subseteq B_\delta(y)$.
To show: If $p \in B_\epsilon(x)$ then $p \in B_\delta(y)$.
Assume $p \in B_\epsilon(x)$.
Since $d(p, y) \le d(p, x) + d(x, y) < \epsilon + d(x, y) < \delta$ then $p \in B_\delta(y)$.
So $B_\epsilon(x) \subseteq B_\delta(y) \subseteq U$.
Since $B_\delta(y) = B$ and $B \in \mathcal{S}$ then $B_\epsilon(x) \subseteq U$.
So $U \in \mathcal{T}$.

$\Rightarrow$: Assume $U \in \mathcal{T}$.
If $x \in U$ then there exists $\epsilon_x \in \mathbb{E}$ such that $B_{\epsilon_x}(x) \subseteq U$.
To show: There exists $\mathcal{S} \subseteq \mathcal{B}$ such that $U = \bigcup_{B\in\mathcal{S}} B$.
Let $\mathcal{S} = \{B_{\epsilon_x}(x) \mid x \in U\}$.
To show: $U = \bigcup_{B\in\mathcal{S}} B$.
To show: (a) $U \supseteq \bigcup_{B\in\mathcal{S}} B$.
(b) $U \subseteq \bigcup_{B\in\mathcal{S}} B$.

(a) If $B \in \mathcal{S}$ then $B = B_{\epsilon_x}(x) \subseteq U$.
So $U \supseteq \bigcup_{B\in\mathcal{S}} B$.
(b) To show: If $x \in U$ then $x \in \left(\bigcup_{B\in\mathcal{S}} B\right)$.
Assume $x \in U$.
Since $x \in B_{\epsilon_x}(x)$ and $B_{\epsilon_x}(x) \in \mathcal{S}$ then $x \in \bigcup_{B\in\mathcal{S}} B$.
So $U \subseteq \left(\bigcup_{B\in\mathcal{S}} B\right)$.

So $U = \bigcup_{B\in\mathcal{S}} B$. □

11.5.2 Interiors and closures

Proposition 11.11. *Let X be a topological space. Let $A \subseteq X$.*

(a) The interior of A is the set of interior points of A.

(b) The closure of A is the set of close points of A.

Proof.

(a) Let $I = \{x \in A \mid x \text{ is an interior point of } A\}$.
To show: $A^\circ = I$.
To show: (aa) $I \subseteq A^\circ$.
(ab) $A^\circ \subseteq I$.

(aa) Let $x \in I$.
Then there exists a neighborhood N of x with $N \subseteq A$.

So there exists an open set U with $x \in U \subseteq N \subseteq A$.
Since $U \subseteq A$ and U is open $U \subseteq A^\circ$.
So $x \in A^\circ$.
So $I \subseteq A^\circ$.

(ab) Assume $x \in A^\circ$.
Then A° is open and $x \in A^\circ \subseteq A$.
So x is a interior point of A.
So $x \in I$.
So $A^\circ \subseteq I$.

So $I = A^\circ$.

(b) Let $C = \{x \in X \mid \text{if } N \in \mathcal{N}(x) \text{ then } N \cap A \neq \emptyset\}$ be the set of close points of A.
Then

$$\begin{aligned} C^c &= \{x \in X \mid \text{there exists } N \in \mathcal{N}(x) \text{ such that } N \cap A = \emptyset\} \\ &= \{x \in X \mid \text{there exists } N \in \mathcal{N}(x) \text{ such that } N \subseteq A^c\}. \end{aligned}$$

which is the set of interior points of A^c.
Thus, by part (a), $C^c = (A^c)^\circ$.
So $C = ((A^c)^\circ)^c$.
To show: $C = \overline{A}$.
To show: $((A^c)^\circ)^c = \overline{A}$.

Claim: If $F \subseteq X$ then $(F^\circ)^c = \overline{F^c}$.
Let $F \subseteq X$.
Then F° is open and $(F^\circ)^c$ is closed.
Since $F^\circ \subseteq F$, then $(F^\circ)^c \supseteq F^c$.
So $(F^\circ)^c \supseteq \overline{F^c}$.
If V is closed and $V \supseteq F^c$ then V^c is open and $V^c \subseteq F$.
Thus, if V is closed and $V \supseteq F^c$ then $V^c \subseteq F^\circ$.
Thus, if V is closed and $V \supseteq F^c$ then $V \supseteq (F^\circ)^c$.
So $(F^\circ)^c = \overline{F^c}$.

Thus $((A^c)^\circ)^c = \overline{(A^c)^c}$.
Thus $C = ((A^c)^\circ)^c = \overline{(A^c)^c} = \overline{A}$.

□

11.5.3 Limits and continuity

Theorem 11.12. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.*
Let $f\colon X \to Y$ be a function.

(a) [Bou, Ch. 1 §2 Theorem 1(d)] f is continuous if and only if f satisfies:

$$\text{if } a \in X \qquad \text{then} \qquad f \text{ is continuous at } a.$$

(b) [Bou, Ch. 1 §7 Prop. 9] Let $a \in X$. Then

$$f \text{ is continuous at } a \qquad \text{if and only if} \qquad \lim_{x \to a} f(x) = f(a).$$

(c) [Bou, Ch. 1 §7 no. 5] Let $a \in X$ such that $a \in \overline{X - \{a\}}$. Then

$$f \text{ is continuous at } a \qquad \text{if and only if} \qquad \lim_{\substack{x \to a \\ x \neq a}} f(x) = f(a).$$

(d) [Bou, Ch. IX §2 no. 7 Proposition 10 and the remark following] Let (X, d) be a strict metric space and let $\mathcal{T}_X$ be the metric space topology on X. Then f is continuous if and only if f satisfies:

if $(x_1, x_2, \ldots)$ is a sequence in X and

$$\text{if } \lim_{n\to\infty} x_n \text{ exists} \qquad \text{then} \qquad \lim_{n\to\infty} f(x_n) = f\Big(\lim_{n\to\infty} x_n\Big).$$

Proof.

(a) $\Rightarrow$: To show: If f is continuous then f satisfies: if $a \in X$ then f is continuous at a.
Assume f is continuous.
To show: If $a \in X$ then f is continuous at a.
Assume $a \in X$.
To show: If $N \in \mathcal{N}(f(a))$ then $f^{-1}(N) \in \mathcal{N}(a)$.
Assume $N \in \mathcal{N}(f(a))$.
Then there exists $V \in \mathcal{T}_Y$ such that $f(a) \in V \subseteq N$.
To show: $f^{-1}(N) \in \mathcal{N}(a)$.
To show: There exists $U \in \mathcal{T}_X$ such that $a \in U \subseteq f^{-1}(N)$.
Let $U = f^{-1}(V)$.
Since f is continuous then U is open in X.
Since $f(a) \in V \subseteq N$ then $a \in f^{-1}(V) = U \subseteq f^{-1}(N)$.
So $f^{-1}(N) \in \mathcal{N}(a)$.
So f is continuous at a.

(a) $\Leftarrow$: Assume that if $a \in X$ then f is continuous at a.
To show: f is continuous.
To show: If $V \in \mathcal{T}_Y$ then $f^{-1}(V) \in \mathcal{T}_X$.
Assume $V \in \mathcal{T}_Y$.
To show: $f^{-1}(V)$ is open in X.
To show: If $a \in f^{-1}(V)$ then a is an interior point of $f^{-1}(V)$.
Assume $a \in f^{-1}(V)$.
To show: There exists $U \in \mathcal{N}(a)$ such that $a \in U \subseteq f^{-1}(V)$.
Since $V \in \mathcal{T}_Y$ and $f(a) \in V$ then $V \in \mathcal{N}(f(a))$.
Since f is continuous at a then $f^{-1}(V) \in \mathcal{N}(a)$.
Let $U = f^{-1}(V)$.
Then $a \in U \subseteq f^{-1}(V)$.
So a is an interior point of $f^{-1}(V)$.
So $f^{-1}(V)$ is open in X.
So f is continuous.

(b) $\Rightarrow$: To show: If f is continuous at a then $\lim_{x\to a} f(x) = f(a)$.
Assume f is continuous at a.
To show: $\lim_{x\to a} f(x) = f(a)$.
To show: If $N \in \mathcal{N}(f(a))$ then there exists $P \in \mathcal{N}(a)$ such that $N \supseteq f(P)$.
Assume $N \in \mathcal{N}(f(a))$.
To show: There exists $P \in \mathcal{N}(a)$ such that $N \supseteq f(P)$.
Since f is continuous at a and $N \in \mathcal{N}(f(a))$ then $f^{-1}(N) \in \mathcal{N}(a)$.
Let $P = f^{-1}(N)$.
Then $f(P) = f(f^{-1}(N)) \subseteq N$.
So $\lim_{x\to a} f(x) = f(a)$.

(b) $\Leftarrow$: To show: If $\lim_{x\to a} f(x) = f(a)$ then f is continuous at a.

Assume $\lim_{x\to a} f(x) = f(a)$.
To show: f is continuous at a.
To show: If $N \in \mathcal{N}(f(a))$ then $f^{-1}(N) \in \mathcal{N}(a)$.
Assume $N \in \mathcal{N}(f(a))$.
To show: $f^{-1}(N) \in \mathcal{N}(a)$.
To show: There exists $U \in \mathcal{T}_X$ such that $a \in U \subseteq f^{-1}(N)$.
Since $\lim_{x\to a} f(x) = f(a)$ then there exists $P \in \mathcal{N}(a)$ such that $N \supseteq f(P)$.
So $f^{-1}(N) \supseteq P$.
Since $P \in \mathcal{N}(a)$, there exists $U \in \mathcal{T}_X$ such that $a \in U \subseteq P$.
So there exists $U \in \mathcal{T}_X$ such that $a \in U \subseteq P \subseteq f^{-1}(N)$.
So $f^{-1}(N) \in \mathcal{N}(a)$.
So f is continuous at a.

(c) $\Rightarrow$: Assume $a \in \overline{X - \{a\}}$.
To show: If f is continuous at a then $\lim_{\substack{x\to a\\ x\neq a}} f(x) = f(a)$.
Assume f is continuous at a.
To show: $\lim_{\substack{x\to a\\ x\neq a}} f(x) = f(a)$.
To show: If $N \in \mathcal{N}(f(a))$ then there exists $P \in \mathcal{N}(a)$ such that $N \supseteq f(P - \{a\})$.
Assume $N \in \mathcal{N}(f(a))$.
To show: There exists $P \in \mathcal{N}(a)$ such that $N \supseteq f(P - \{a\})$.
Since f is continuous at a and $N \in \mathcal{N}(f(a))$ then $f^{-1}(N) \in \mathcal{N}(a)$.
Let $P = f^{-1}(N)$.
Then $f(P - \{a\}) \subseteq f(P) = f(f^{-1}(N)) \subseteq N$.
So $\lim_{\substack{x\to a\\ x\neq a}} f(x) = f(a)$.

(c) $\Leftarrow$: Assume $a \in \overline{X - \{a\}}$.
To show: If $\lim_{\substack{x\to a\\ x\neq a}} f(x) = f(a)$ then f is continuous at a.
Assume $\lim_{\substack{x\to a\\ x\neq a}} f(x) = f(a)$.
To show: f is continuous at a.
To show: If $N \in \mathcal{N}(f(a))$ then $f^{-1}(N) \in \mathcal{N}(a)$.
Assume $N \in \mathcal{N}(f(a))$.
To show: $f^{-1}(N) \in \mathcal{N}(a)$.
To show: There exists $U \in \mathcal{T}_X$ such that $a \in U \subseteq f^{-1}(N)$.
Since $\lim_{\substack{x\to a\\ x\neq a}} f(x) = f(a)$ there exists $P \in \mathcal{N}(a)$ such that $N \supseteq f(P - \{a\})$.
So $f^{-1}(N) \supseteq P - \{a\}$.
Since $N \in \mathcal{N}(f(a))$ then $f(a) \in N$ and $a \in f^{-1}(N)$.
So $f^{-1}(N) \supseteq P$.
Since $P \in \mathcal{N}(a)$, there exists $U \in \mathcal{T}_X$ such that $a \in U \subseteq P$.
So there exists $U \in \mathcal{T}_X$ such that $a \in U \subseteq P \subseteq f^{-1}(N)$.
So $f^{-1}(N) \in \mathcal{N}(a)$.
So f is continuous at a.

(d) $\Rightarrow$: Assume f is continuous.
To show: f satisfies

$$\begin{array}{c}\text{if } (x_1, x_2, \ldots) \text{ is a sequence in } X \text{ and } \lim_{n\to\infty} x_n \text{ exists} \\ \text{then} \quad f\Big(\lim_{n\to\infty} x_n\Big) = \lim_{n\to\infty} f(x_n).\end{array} \tag{*}$$

Assume $(x_1, x_2, \ldots)$ is a sequence in X and $\lim_{n\to\infty} x_n = a$.
To show: $f(a) = \lim_{n\to\infty} f(x_n)$.
To show: If $N \in \mathcal{N}(f(a))$ then there exists $t \in \mathbb{Z}_{>0}$ such that $N \supseteq (f(x_t), f(x_{t+1}), \ldots)$.
Assume $N \in \mathcal{N}(f(a))$.
Since f is continuous then $f^{-1}(N) \in \mathcal{N}(a)$.
Since $\lim_{n\to\infty} x_n = a$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that $f^{-1}(N) \supseteq \{x_\ell, x_{\ell+1}, \ldots\}$.
Let $t = \ell$.
Then $f^{-1}(N) \supseteq \{x_t, x_{t+1}, \ldots\}$.
So $N \supseteq \{f(x_t), f(x_{t+1}), \ldots\}$.
So f satisfies (*).

(d) $\Leftarrow$: To show: If f is not continuous then f does not satisfy (*).
Assume f is not continuous.
Then there exists a such that f is not continuous at a.
So there exists $N \in \mathcal{N}(f(a))$ such that $f^{-1}(N) \notin \mathcal{N}(a)$.
To show: There exists a sequence $(x_1, x_2, \ldots)$ such that $\lim_{n\to\infty} x_n$ exists and $\lim_{n\to\infty} f(x_n) \neq f\big(\lim_{n\to\infty} x_n\big)$.
Since $f^{-1}(N) \notin \mathcal{N}(a)$ then $f^{-1}(N) \not\supseteq B_{10^{-\ell}}(a)$, for $\ell \in \mathbb{Z}_{>0}$. Let

$$x_1 \in B_{10^{-1}}(a) \cap f^{-1}(N)^c, \quad x_2 \in B_{10^{-2}}(a) \cap f^{-1}(N)^c, \quad \ldots.$$

To show: (da) $\lim_{n\to\infty} x_n = a$.
(db) $\lim_{n\to\infty} f(x_n) \neq f(a)$.

(da) To show: If $P \in \mathcal{N}(a)$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell}$ then $x_n \in P$.
Assume $P \in \mathcal{N}(a)$.
To show: There exists $\ell \in \mathbb{Z}_{>0}$ such that $P \supseteq \{x_\ell, x_{\ell+1}, \ldots\}$.
Since $P \in \mathcal{N}(a)$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that $P \supseteq B_{10^{-\ell}}(a)$.
To show: $P \supseteq \{x_\ell, x_{\ell+1}, \ldots\}$.
To show: If $n \in \mathbb{Z}_{\geq \ell}$ then $x_n \in P$.
Assume $n \in \mathbb{Z}_{\geq \ell}$.
Since $n \geq \ell$ then $10^{-\ell} \leq 10^{-n}$ and $x_n \in B_{10^{-n}}(a) \subseteq B_{10^{-\ell}}(a) \subseteq P$.
So $P \supseteq \{x_\ell, x_{\ell+1}, \ldots\}$.
So $\lim_{n\to\infty} x_n = a$.

(db) To show: $\lim_{n\to\infty} f(x_n) \neq f(a)$.
To show: There exists $M \in \mathcal{N}(f(a))$ such that $\{j \in \mathbb{Z}_{>0} \mid f(x_j) \in M^c\}$ is infinite.
Let $M = N$.
To show: $\{j \in \mathbb{Z}_{>0} \mid f(x_j) \in N^c\}$ is infinite.
Since $x_j \in f^{-1}(N)^c$ then $f(x_j) \notin N$, for $j \in \mathbb{Z}_{>0}$.
So $\{f(x_1), f(x_2), \ldots\} \subseteq N^c$.
So $\{j \in \mathbb{Z}_{>0} \mid f(x_j) \in N^c\}$ is infinite.
So $\lim_{n\to\infty} f(x_n) \neq f(a)$.

So f does not satisfy (*).

□

To change the proof of (d) above to a proof for first countable topological spaces $(X, \mathcal{T}_X)$, replace the use of the open balls $B_{10^{-1}}(a) \supseteq B_{10^{-2}}(a) \supseteq \cdots$ by generators $B_1 \supseteq B_2 \supseteq \cdots$ of $\mathcal{N}(a)$, the neighborhood filter of a.

11.5.4 The topology in a metric space is determined by limits of sequences

Theorem 11.13. *Let (X,d) be a strict metric space and let $A \subseteq X$ and let $\overline{A}$ be the closure of A. Then*

$$\overline{A} = \{z \in X \mid \text{there exists a sequence } (a_1, a_2, \ldots) \text{ in } A \text{ with } z = \lim_{n\to\infty} a_n\}.$$

Proof. Let $R = \{z \in X \mid \text{there exists a sequence } (a_1, a_2, \ldots) \text{ in } A \text{ with } z = \lim_{n\to\infty} a_n\}$.

To show: (a) $R \subseteq \overline{A}$.
(b) $\overline{A} \subseteq R$.

(a) To show: If $z \in R$ then $z \in \overline{A}$.
Assume $z \in R$.
To show: $z \in \overline{A}$.
We know there exists a sequence $(a_1, a_2, \ldots)$ in A with $z = \lim_{n\to\infty} a_n$.
To show: z is a close point of A.
To show: If N is a neighborhood of z then $N \cap A \neq \emptyset$.
Assume N is a neighborhood of z.
Since $\lim_{n\to\infty} a_n = z$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell}$ then $a_n \in N$.
So $N \cap A \neq \emptyset$.
So z is a close point of A.
So $R \subseteq \overline{A}$.

(b) To show: $\overline{A} \subseteq R$.
To show: If $z \in \overline{A}$ then $z \in R$.
Let $z \in \overline{A}$.
To show: $z \in R$.
To show: There exists a sequence $(a_1, a_2, \ldots)$ in A with $z = \lim_{n\to\infty} a_n$.
Using that z is a close point of A,

$$\text{let} \quad a_1 \in B_{0.1}(z) \cap A, \quad a_2 \in B_{0.01}(z) \cap A, \quad a_3 \in B_{0.001}(z) \cap A, \quad \ldots.$$

To show: $z = \lim_{n\to\infty} a_n$.
To show: If P is a neighborhood of z then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell}$ then $a_n \in P$.
Let P be a neighborhood of z.
Then there exists $\ell \in \mathbb{Z}_{>0}$ such that $B_{10^{-\ell}}(z) \subseteq P$.
To show: If $n \in \mathbb{Z}_{\geq \ell}$ then $a_n \in P$.
Assume $n \in \mathbb{Z}_{\geq \ell}$.
Since $n \geq \ell$ then $10^{-n} \leq 10^{-\ell}$ and

$$a_n \in B_{10^{-n}}(z) \subseteq B_{10^{-\ell}}(z) \subseteq P,$$

So $\lim_{n\to\infty} a_n = z$.
So $z \in R$.
So $\overline{A} \subseteq R$.

□

To change the proof of (b) above to a proof for first countable topological spaces $(X, \mathcal{T}_X)$, replace the use of the open balls $B_{10^{-1}}(a) \supseteq B_{10^{-2}}(a) \supseteq \cdots$ by generators $B_1 \supseteq B_2 \supseteq \cdots$ of $\mathcal{N}(a)$, the neighborhood filter of a.

11.5.5 Limits in metric spaces

Proposition 11.14. *Let (X, d_X) and (Y, d_Y) be strict metric spaces, let $\mathcal{T}_X$ be the metric space topology on X and let $\mathcal{T}_Y$ be the metric space topology on Y. Let $f\colon X \to Y$ be a function and let $y \in Y$.*

(a) Let $a \in X$. Then $\lim_{x\to a} f(x) = y$ if and only if f satisfies

$$\text{if } \epsilon \in \mathbb{E} \text{ then there exists } \delta \in \mathbb{E} \text{ such that}$$
$$\text{if } x \in X \text{ and } d_X(x,a) < \delta \quad \text{then} \quad d_Y(f(x), y) < \epsilon.$$

(b) Let $a \in X$ such that $a \in \overline{X - \{a\}}$. Then $\lim_{\substack{x\to a\\ x\ne a}} f(x) = y$ if and only if f satisfies

$$\text{if } \epsilon \in \mathbb{E} \text{ then there exists } \delta \in \mathbb{E} \text{ such that}$$
$$\text{if } x \in X \text{ and } 0 < d_X(x,a) < \delta \quad \text{then} \quad d_Y(f(x), y) < \epsilon.$$

(c) Let $(x_1, x_2, \ldots)$ be a sequence in X and let $z \in X$. Then $\lim_{n\to\infty} x_n = z$ if and only if $(x_1, x_2, \ldots)$ satisfies

$$\text{if } \varepsilon \in \mathbb{E} \text{ then there exists } \ell \in \mathbb{Z}_{>0} \text{ such that} \quad \text{if } n \in \mathbb{Z}_{\ge \ell} \text{ then } d(x_n, z) < \varepsilon.$$

Proof. (a) By definition, $\lim_{x\to a} f(x) = y$ if and only if f satisfies: if $N \in \mathcal{N}(y)$ then there exists $P \in \mathcal{N}(a)$ such that $N \supseteq f(P)$.

By definition of the metric space topology, $N \in \mathcal{N}(y)$ if and only if there exists $\epsilon \in \mathbb{E}$ such that $B_\epsilon(y) \subseteq N$.

Thus $\lim_{x\to a} f(x) = y$ if and only if f satisfies: if $B_\epsilon(y)$ is an open ball at y then there exists $B_\delta(a)$, an open ball at a such that $B_\epsilon(y) \supseteq f(B_\delta(a))$.

By definition, $B_\delta(a) = \{x \in X \mid d(x,a) < \delta\}$.

Thus, $\lim_{x\to a} f(x) = y$ if and only if f satisfies: if $\varepsilon \in \mathbb{E}$ then there exists $\delta \in \mathbb{E}$ such that if $x \in X$ and $d_X(x,a) < \delta$ then $d_Y(f(x), y) < \varepsilon$.

(b) By definition, $\lim_{\substack{x\to a\\ x\ne a}} f(x) = y$ if and only if f satisfies: if $N \in \mathcal{N}(y)$ then there exists $P \in \mathcal{N}(a)$ such that $N \supseteq f(P - \{a\})$.

By definition of the metric space topology, $N \in \mathcal{N}(y)$ if and only if there exists $\epsilon \in \mathbb{E}$ such that $B_\epsilon(y) \subseteq N$.

Thus $\lim_{\substack{x\to a\\ x\ne a}} f(x) = y$ if and only if f satisfies: if $B_\epsilon(y)$ is an open ball at y then there exists $B_\delta(a)$, an open ball at a such that $B_\epsilon(y) \supseteq f(B_\delta(a) - \{a\})$.

By definition, $B_\epsilon(y) = \{x \in Y \mid d(x,y) < \epsilon\}$ and $B_\delta(a) - \{a\} = \{x \in X \mid 0 < d(x,a) < \delta\}$.

Thus, $\lim_{\substack{x\to a\\ x\ne a}} f(x) = y$ if and only if f satisfies: if $\epsilon \in \mathbb{E}$ then there exists $\delta \in \mathbb{E}$ such that if $x \in X$ and $0 < d_X(x,a) < \delta$ then $d_Y(f(x), y) < \epsilon$.

(c) By definition, $\lim_{n\to\infty} x_n = z$ if and only if $(x_1, x_2, \ldots)$ satisfies: if $P \in \mathcal{N}(z)$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that $P \supseteq \{x_\ell, x_{\ell+1}, \ldots\}$.

By definition of the metric space topology, $P \in \mathcal{N}(y)$ if and only if there exists $\epsilon \in \mathbb{E}$ such that $B_\epsilon(y) \subseteq P$.

So $\lim_{n\to\infty} x_n = z$ if and only if $(x_1, x_2, \ldots)$ satisfies: if $\epsilon \in \mathbb{E}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that $B_\epsilon(z) \supseteq \{x_\ell, x_{\ell+1}, \ldots, \}$.

By definition, $B_\epsilon(a) = \{x \in X \mid d(x,a) < \epsilon\}$.
Thus, $\lim_{n\to\infty} x_n = z$ if and only if $(x_1, x_2, \ldots)$ satisfies: if $\epsilon \in \mathbb{E}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell}$ then $d(x_n, z) < \epsilon$. $\square$