7 Limits and continuity: Review from Calculus 2

7.1 Limits

The *tolerance set* is

$$\mathbb{E} = \{10^{-1}, 10^{-2}, \ldots\}.$$

Let (X, d) be a metric space and let $f\colon X \to \mathbb{R}$. Let $a \in X$ and $\ell \in \mathbb{R}$.

$$\lim_{x\to a} f(x) = \ell \qquad \text{means}$$

$$\text{if } \varepsilon \in \mathbb{E} \text{ then there exists } \delta \in \mathbb{E} \text{ such that} \quad \text{if } d(x,a) < \delta \text{ then } d(f(x), \ell) < \varepsilon.$$

Here is a translation into the language of "English":

In English	In Math
The client has a machine f that produces steel rods of length ℓ for sales.	Let $f\colon X \to \mathbb{R}$ and let $\ell \in \mathbb{R}$.
The output of f gets closer and closer to ℓ as the input gets closer and closer to a means	$\lim_{x\to a} f(x) = \ell$ means
if you give me a tolerance the client needs, in other words, the number of decimal places of accuracy the client requires	if $\varepsilon \in \mathbb{E}$
then my business will tell you	then there exists
the accuracy you need on the dials of the machine so that	$\delta \in \mathbb{E}$ such that
if the dials are set within δ of a	if $d(x,a) < \delta$
then the output of the machine will be within ε of ℓ.	then $d(f(x), \ell) < \epsilon$.

Let (X, d) be a metric space and let $a_1, a_2, \ldots$ be a sequence in X. Let $\ell \in \mathbb{R}$.

$$\lim_{n\to\infty} a_n = \ell \qquad \text{means}$$

$$\text{if } \varepsilon \in \mathbb{E} \text{ then there exists } N \in \mathbb{Z}_{>0} \text{ such that} \quad \text{if } n \in \mathbb{Z}_{\geq N} \text{ then } d(a_n, \ell) < \varepsilon.$$

7.2 Continuity

Let $p \in \mathbb{R}^m$. A function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is *continuous at p* if

$$\lim_{x\to p} f(x) = f(p).$$

7.3 Limits and continuity results

7.3.1 x^n and e^x are continuous

Proposition 7.1.

(a) *Let $n \in \mathbb{Z}_{>0}$. The function $f\colon \mathbb{C} \to \mathbb{C}$ given by $f(x) = x^n$ is continuous.*

(b) *The function $f\colon \mathbb{C} \to \mathbb{C}$ given by $f(x) = e^x$ is continuous.*

7.3.2 Behavior of x^n as $n \in \mathbb{Z}_{>0}$ gets large

HW: Let $x \in \mathbb{C}$. Show that

$$\lim_{n\to\infty} x^n = \begin{cases} 0, & \text{if } |x| < 1, \\ \text{diverges in } \mathbb{C}, & \text{if } |x| > 1, \\ 1, & \text{if } x = 1, \\ \text{diverges in } \mathbb{C}, & \text{if } |x| = 1 \text{ and } x \neq 1. \end{cases}$$

7.3.3 Behavior of $1 + x + x^2 + \cdots + x^n$ as $n \in \mathbb{Z}_{>0}$ gets large

HW: Let $x \in \mathbb{C}$. Show that

$$\lim_{n\to\infty} (1 + x + x^2 + \cdots + x^n) = \lim_{n\to\infty} \frac{1 - x^{n+1}}{1 - x} = \begin{cases} \dfrac{1}{1-x}, & \text{if } |x| < 1, \\ \text{diverges in } \mathbb{C}, & \text{if } |x| \geq 1. \end{cases}$$

For example, if $x = \frac{1}{2}$ then

$$\lim_{n\to\infty} \left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots + \left(\frac{1}{2}\right)^n\right) = \lim_{n\to\infty} \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} = \frac{1}{1 - \frac{1}{2}} = 2.$$

7.3.4 Favorite limits

Proposition 7.2. *(a) If $n \in \mathbb{Z}_{>0}$ then, in $\mathbb{R}$, $\lim\limits_{x\to\infty} x^n e^{-x} = 0$.*

(b) If $\alpha \in \mathbb{R}_{>0}$ then $\lim\limits_{x\to\infty} x^{-\alpha} \log x = 0$.

(c) Let $p \in \mathbb{R}_{>0}$. Then $\lim\limits_{n\to\infty} \dfrac{1}{n^p} = 0$.

(d) Let $p \in \mathbb{R}_{>0}$. Then $\lim\limits_{n\to\infty} p^{1/n} = 0$.

(e) $\lim\limits_{n\to\infty} n^{1/n} = 1$.

7.4 Limits and continuity proofs

7.4.1 x^n and e^x are continuous

Proposition 7.3.

(a) *Let $n \in \mathbb{Z}_{>0}$. The function $f\colon \mathbb{C} \to \mathbb{C}$ given by $f(x) = x^n$ is continuous.*

(b) *The function $f\colon \mathbb{C} \to \mathbb{C}$ given by $f(x) = e^x$ is continuous.*

Proof. Assume $n \in \mathbb{Z}_{>0}$.
To show: If $a \in \mathbb{C}$ then $\lim\limits_{x\to a} x^n = a^n$.
Assume $a \in \mathbb{C}$.
To show $\lim\limits_{x\to a} x^n = a^n$.
Using Theorem 7.6,
To show: $\lim\limits_{y\to 0} (y+a)^n = a^n$.
Using Theorem 7.5(a)
To show: $\lim\limits_{y\to 0} |(y+a)^n - a^n| = 0$.

$$\begin{aligned}
\lim_{y\to 0} |(y+a)^n - a^n| &= \lim_{y\to 0} |y^n + ny^{n-1}a + \cdots + na^{n-1}y + a^n - a^n| \\
&= \lim_{y\to 0} |y^n + ny^{n-1}a + \cdots + na^{n-1}y| \\
&= \lim_{y\to 0} |y(y^{n-1} + ny^{n-2}a + \cdots + na^{n-1})| \\
&= \lim_{y\to 0} |y|\,|(y^{n-1} + ny^{n-2}a + \cdots + na^{n-1})| \\
&\le \lim_{y\to 0} |y| \left(|y|^{n-1} + n|y|^{n-2}|a| + \cdots + n|a|^{n-1}\right) && \text{(triangle inequality for } |\ |) \\
&\le \lim_{y\to 0} |y| \cdot n|a|^{n-1} && \text{(by using Theorem 7.7(b))} \\
&= 0 \cdot n|a|^{n-1} = 0. && \text{(by using Theorem 7.5(b))} \\
&= 0.
\end{aligned}$$

So $f(x) = x^n$ is continuous at a.

(An alternative proof (sketch) is that

(1) $f(x) = x$ (the identity function) is continuous,

(2) the product is continuous (since $\mathbb{C}$ is a topological field),

and therefore, by induction, if $a \in \mathbb{C}$ then $\lim\limits_{x\to a} x^n = a^n$.)

(b) To show: If $a \in \mathbb{C}$ then $\lim\limits_{x\to a} e^x = e^a$.
Assume $a \in \mathbb{C}$.
To show: $\lim\limits_{x\to a} e^x = e^a$.

Case 1: $a = 0$. To show: $\lim\limits_{x\to 0} e^x = e^0$.
Using Theorem 7.5(a), To show $\lim\limits_{x\to 0} |e^x - 1| = 0$.

$$\begin{aligned}
\lim_{x\to 0} |e^x - 1| &= \lim_{x\to 0} \left| \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - 1 \right| \\
&= \lim_{x\to 0} \left| x\left(1 + x + \frac{x}{2!} + \frac{x^2}{3!} + \cdots\right) \right| \\
&\le \lim_{x\to 0} |x| \left(1 + |x| + \frac{|x|}{2!} + \frac{|x|^2}{3!} + \cdots\right) && \text{(triangle inequality for } |\ |) \\
&\le \lim_{x\to 0} |x| \left(1 + |x| + |x| + |x|^2 + \cdots\right) && \text{(by Theorem 7.7)} \\
&= \lim_{x\to 0} |x| \frac{1}{1-|x|} = 0 \cdot 1 && \text{(by Theorem 7.5(b))} \\
&= 0.
\end{aligned}$$

Case 2: $a \neq 0$. To show $\lim_{x\to a} e^x = e^a$.

$$\begin{aligned}
\lim_{x\to a} e^x &= \lim_{y\to 0} e^{y+a} && \text{(by Theorem 7.6)} \\
&= \lim_{y\to 0} e^a e^y = e^a \lim_{y\to 0} e^y && \text{(by Theorem 7.5(b))} \\
&= e^a \cdot e^0 && \text{(by Case 1)} \\
&= e^{a+0} = e^a.
\end{aligned}$$

So e^x is continuous at $x = a$. □

7.4.2 Behavior of x^n as $n \in \mathbb{Z}_{>0}$ gets large

HW: Let $x \in \mathbb{C}$. Show that

$$\lim_{n\to\infty} x^n = \begin{cases} 0, & \text{if } |x| < 1, \\ \text{diverges in } \mathbb{C}, & \text{if } |x| > 1, \\ 1, & \text{if } x = 1, \\ \text{diverges in } \mathbb{C}, & \text{if } |x| = 1 \text{ and } x \neq 1. \end{cases}$$

Proof. Let $x \in \mathbb{C}$.
Case $|x| < 1$. To show: $\lim_{n\to\infty} x^n = 0$.
Let $N \in \mathbb{Z}_{>0}$ such that $|x| < 1 - \frac{1}{N+1}$. Then

$$\begin{aligned}
\lim_{n\to\infty} |x^n - 0| &= \lim_{n\to\infty} |x|^n \le \lim_{n\to\infty} \left(1 - \frac{1}{N+1}\right)^n \\
&= \lim_{n\to\infty} \left(\frac{N+1-1}{N+1}\right)^n = \lim_{n\to\infty} \left(\frac{N}{N+1}\right)^n = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{N})^n} \\
&= \lim_{n\to\infty} \frac{1}{1 + n\frac{1}{N} + \cdots + (\frac{1}{N})^n} \\
&\le \lim_{n\to\infty} \frac{1}{1+\frac{n}{N}} = \lim_{n\to\infty} \frac{N}{n+N} = N \cdot \lim_{n\to\infty} \frac{1}{n+N} = N \cdot 0 = 0.
\end{aligned}$$

Case $|x| > 1$. To show: $\lim_{n\to\infty} x^n$ diverges in $\mathbb{C}$.

Let $N \in \mathbb{Z}_{>0}$ such that $|x| > 1 - \frac{1}{N}$. Then

$$|x|^n > \left(1 + \frac{1}{N}\right)^n = 1 + n\left(\frac{1}{N}\right) + \cdots \left(\frac{1}{N}\right)^n > \left(\frac{1}{N}\right) n.$$

Since $\left(\frac{1}{N}\right) n$ is unbounded as n gets larger and larger then $|x|^n$ is unbounded as $n \to \infty$. So $\lim_{n\to\infty} x^n$ diverges in $\mathbb{C}$.

Case $x = 1$. In this case $(x, x^2, x^3, x^4, \ldots) = (1, 1^2, 1^3, 1^4, \ldots) = (1, 1, 1, 1, \ldots)$. So $\lim\limits_{n\to\infty} x^n = \lim\limits_{n\to\infty} 1^n = \lim\limits_{n\to\infty} 1 = 1$.

Case $|x| = 1$ *and* $x \neq 1$. Then $x = e^{i\theta}$ with $\theta \in \mathbb{R}_{(0,2\pi)}$. FINISH THE PROOF to show that this case diverges in $\mathbb{C}$. □

7.4.3 Behavior of $1 + x + x^2 + \cdots + x^n$ as $n \in \mathbb{Z}_{>0}$ gets large

HW: Let $x \in \mathbb{C}$. Show that

$$\lim_{n\to\infty} (1 + x + x^2 + \cdots + x^n) = \lim_{n\to\infty} \frac{1 - x^{n+1}}{1 - x} = \begin{cases} \dfrac{1}{1-x}, & \text{if } |x| < 1, \\ \text{diverges in } \mathbb{C}, & \text{if } |x| \geq 1. \end{cases}$$

For example, if $x = \frac{1}{2}$ then

$$\lim_{n\to\infty} \left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots + \left(\frac{1}{2}\right)^n\right) = \lim_{n\to\infty} \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} = \frac{1}{1 - \frac{1}{2}} = 2.$$

7.4.4 Favorite limits

Proposition 7.4. *(a) If $n \in \mathbb{Z}_{>0}$ then, in $\mathbb{R}$, $\lim\limits_{x\to\infty} x^n e^{-x} = 0$.*

(b) If $\alpha \in \mathbb{R}_{>0}$ then $\lim\limits_{x\to\infty} x^{-\alpha} \log x = 0$.

(c) Let $p \in \mathbb{R}_{>0}$. Then $\lim\limits_{n\to\infty} \dfrac{1}{n^p} = 0$.

(d) Let $p \in \mathbb{R}_{>0}$. Then $\lim\limits_{n\to\infty} p^{1/n} = 0$.

(e) $\lim\limits_{n\to\infty} n^{1/n} = 1$.

Proof sketches. (a) Let $n \in \mathbb{Z}_{>0}$. Then, in $\mathbb{R}$,

$$0 \leq \lim_{x\to\infty} x^n e^{-x} = \lim_{x\to\infty} \frac{x^n}{e^x} \leq \lim_{x\to\infty} \frac{x^n}{\frac{1}{(n+1)!} x^{n+1}} = \lim_{x\to\infty} \frac{(n+1)!}{x} = (n+1)! \lim_{x\to\infty} \frac{1}{x} = 0.$$

(b) Let $\alpha \in \mathbb{R}_{>0}$ and let $\epsilon \in \mathbb{E}$ with $\varepsilon < \alpha$. Then

$$\begin{aligned} 0 &\leq \lim_{x\to\infty} x^{-\alpha} \log x = \lim_{x\to\infty} \big(x^{-\alpha} \int_1^x \frac{1}{t} dt\big) \\ &\leq \lim_{x\to\infty} \big(x^{-\alpha} \int_1^x t^{\varepsilon - 1} dt\big) = \lim_{x\to\infty} x^{-\alpha} \left(\frac{x^\varepsilon - 1^\varepsilon}{\varepsilon}\right) \\ &= \lim_{x\to\infty} \left(\frac{x^{\varepsilon-\alpha} - x^{-\alpha}}{\varepsilon}\right) \leq \lim_{x\to\infty} \frac{x^{\varepsilon-\alpha}}{\varepsilon} = 0. \end{aligned}$$

(c) To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{>N}$ then $\left|\frac{1}{n^p}\right| < \varepsilon$.
Assume $\varepsilon \in \mathbb{R}_{>0}$.

Let N be an integer greater than $\left(\dfrac{1}{\varepsilon}\right)^{\frac{1}{p}}$.

To show: If $n \in \mathbb{Z}_{>N}$ then $\left|\frac{1}{n^p}\right| < \varepsilon$.
Assume $n \in \mathbb{Z}_{>N}$.
To show: $\left|\frac{1}{n^p}\right| < \varepsilon$.

$$\left|\frac{1}{n^p}\right| \le \left|\frac{1}{N^p}\right| = \frac{1}{\left(\left(\frac{1}{\varepsilon}\right)^{\frac{1}{p}}\right)^p} = \frac{1}{\left(\frac{1}{\varepsilon}\right)} = \varepsilon.$$

(d)

$$\lim_{n\to\infty} p^{1/n} = \lim_{n\to\infty} (e^{\log p})^{1/n} = \lim_{n\to\infty} e^{\frac{1}{n}\log p} = e^{\log p \lim_{n\to\infty}\frac{1}{n}} = e^{\log p \cdot 0} = e^0 = 1.$$

(e) To show: $\lim\limits_{n\to\infty} n^{1/n} = 1$.

To show: $\lim\limits_{n\to\infty} \left(n^{1/n} - 1\right) = 0$.
We know:

$$\begin{aligned} n &= \left(n^{1/n}\right)^n = \left(\left(n^{1/n}-1\right)+1\right)^n \\ &= 1 + n\left(n^{1/n}-1\right) + \frac{n(n-1)}{2}\left(n^{1/n}-1\right)^2 + \cdots \ge \frac{n(n-1)}{2}\left(n^{1/n}-1\right)^2. \end{aligned}$$

So $\left(n^{1/n}-1\right)^2 \le \dfrac{2}{n-1}$.

So $0 \le \lim\limits_{n\to\infty}\left(n^{1/n}-1\right)^2 \le \lim\limits_{n\to\infty}\dfrac{2}{n-1} = 0$.

So $\lim\limits_{n\to\infty}\left(n^{1/n}-1\right) = 0$. $\square$

7.4.5 The interest sequence

7.4.6 Picard iteration

7.5 Limits and addition, scalar multiplication, multiplication, composition and order

The *tolerance set* is
$$\mathbb{E} = \{10^{-1}, 10^{-2}, \ldots\}.$$
Let (X,d) be a metric space and let $f\colon X \to \mathbb{R}$. Let $a \in X$ and $\ell \in \mathbb{R}$.
$$\lim_{x\to a} f(x) = \ell \qquad \text{means}$$
$$\text{if } \varepsilon \in \mathbb{E} \text{ then there exists } \delta \in \mathbb{E} \text{ such that} \quad \text{if } d(x,a) < \delta \text{ then } d(f(x),\ell) < \varepsilon.$$

Let (X,d) be a metric space and let $a_1, a_2, \ldots$ be a sequence in X. Let $\ell \in \mathbb{R}$.
$$\lim_{n\to\infty} a_n = \ell \qquad \text{means}$$
$$\text{if } \varepsilon \in \mathbb{E} \text{ then there exists } N \in \mathbb{Z}_{>0} \text{ such that} \quad \text{if } n \in \mathbb{Z}_{\geq N} \text{ then } d(a_n,\ell) < \varepsilon.$$

Theorem 7.5. *Let (X,d) be a metric space. Let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be functions and let $a \in X$.*
$$\textit{Assume that} \quad \lim_{x\to a} f(x) \quad \textit{and} \quad \lim_{x\to a} g(x) \qquad \textit{exist.}$$
Then

(a) $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$,

(b) If $c \in \mathbb{R}$ *then* $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$,

(c) $\lim_{x\to a}(f(x)g(x)) = \big(\lim_{x\to a} f(x)\big)\big(\lim_{x\to a} g(x)\big)$.

Theorem 7.6. *Let (X,d) and (Y,ρ) and (Z,σ) be metric spaces. Let $f\colon X \to Z$ and $g\colon X \to Y$ be functions and let $a \in X$ and $\ell \in Y$.*
$$\textit{Assume that} \quad \lim_{x\to a} g(x) \textit{ and } \lim_{x\to a} f(g(x)) \textit{ exist} \qquad \textit{and} \qquad \lim_{x\to a} g(x) = \ell.$$
Then
$$\lim_{y\to\ell} f(y) = \lim_{x\to a} f(g(x)).$$

Theorem 7.7.

(a) Let $(a_1, a_2, \ldots)$ and $(b_1, b_2, \ldots)$ be sequences in $\mathbb{R}$.
Assume that $\lim_{n\to\infty} a_n$ *and* $\lim_{n\to\infty} b_n$ *exist and*
$$\textit{if } n \in \mathbb{Z}_{>0} \textit{ then } a_n \leq b_n. \qquad \textit{Then} \qquad \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n.$$

(b) Let (X,d) be a metric space and let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be functions. Let $a \in X$.
Assume that $\lim_{x\to a} f(x)$ *and* $\lim_{x\to a} g(x)$ *exist and*
$$\textit{if } x \in X \textit{ then } f(x) \leq g(x). \qquad \textit{Then} \qquad \lim_{x\to a} f(x) \leq \lim_{x\to a} g(x).$$

7.6 Limits and operations proofs

Theorem 7.8. (Limits and addition, scalar multiplication and multiplication) *Let (X, d) be a metric space. Let $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ be functions and let $a \in X$.*

$$\text{Assume that} \quad \lim_{x\to a} f(x) \quad \text{and} \quad \lim_{x\to a} g(x) \qquad \text{exist.}$$

Then

(a) $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$,

(b) *If* $c \in \mathbb{R}$ *then* $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$,

(c) $\lim_{x\to a}(f(x)g(x)) = \big(\lim_{x\to a} f(x)\big)\big(\lim_{x\to a} g(x)\big)$.

Proof.

(a) Let $l_1 = \lim_{x\to a} f(x)$ and $l_2 = \lim_{x\to a} g(x)$.
To show: $\lim_{x\to a}(f(x)+g(x)) = l_1 + l_2$.
To show: If $\epsilon \in \mathbb{E}$ then there exists $\delta \in \mathbb{E}$ such that if $x \in B_\delta(a)$ then $|(f(x)+g(x))-(l_1+l_2)| < \epsilon$.
Assume $\epsilon \in \mathbb{E}$.
We know: There exists $\delta_1 \in \mathbb{E}$ such that if $x \in B_{\delta_1}(a)$ then $|f(x) - l_1| < \frac{\epsilon}{2}$.
We know: There exists $\delta_2 \in \mathbb{E}$ such that if $x \in B_{\delta_2}(a)$ then $|f(x) - l_2| < \frac{\epsilon}{2}$.
Let $\delta = \min(\delta_1, \delta_2)$.
To show: $|(f(x) + g(x)) - (l_1 + l_2)| < \epsilon$.

$$\begin{aligned}|(f(x)+g(x)) - (l_1+l_2)| &= |(f(x)-l_1) + (g(x)-l_2)| \\ &\leq |f(x)-l_1| + |g(x)-l_2| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

So $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.

(b) Let $l = \lim_{x\to a} f(x)$.
To show: $\lim_{x\to a} cf(x) = cl$.
To show: If $\epsilon \in \mathbb{E}$ then there exists $\delta \in \mathbb{E}$ such that if $x \in X$ and $d(x, a) < \delta$ then $d(cf(x), cl) < \epsilon$.
Assume $\epsilon \in \mathbb{E}$.
We know: There exists $\delta \in \mathbb{E}$ such that if $x \in X$ and $d(x, a) < \delta$ then $d(f(x), l) < \frac{\epsilon}{|c|}$.
To show: If $x \in X$ and $d(x, a) < \delta$ then $d(cf(x), cl) < \epsilon$.
Assume $x \in X$ and $d(x, a) < \delta$.
To show: $d(cf(x), cl) < \epsilon$.

$$d(cf(x), cl) = |cf(x) - cl| = |c| \cdot |f(x) - l| < |c| \cdot \frac{\epsilon}{|c|} = \epsilon.$$

So $\lim_{x\to a} cf(x) = cl$.

(c) Let $l_1 = \lim_{x\to a} f(x)$ and $l_2 = \lim_{x\to a} g(x)$.

To show: $\lim_{x\to a}(f(x)g(x)) = l_1l_2$.
To show: If $\epsilon \in \mathbb{E}$ then there exists $\delta \in \mathbb{E}$ such that if $x \in X$ and $x \in B_\delta(a)$ then $|f(x)g(x)-l_1l_2| < \epsilon$.
Assume $\epsilon \in \mathbb{E}$.
Let $\epsilon_1 = \min\left(\frac{\epsilon}{|\ell_1|+|\ell_2|+1}, 1\right)$.
Since $\lim_{x\to a} f(x) = l_1$, there exists $\delta_1 \in \mathbb{E}$ such that if $x \in X$ and $d(x,a) < \delta_1$ then $|f(x)-l_1| < \epsilon_1$.
Since $\lim_{x\to a} f(x) = l_2$, there exists $\delta_2 \in \mathbb{E}$ such that if $x \in X$ and $d(x,a) < \delta_2$ then $|f(x)-l_2| < \epsilon_1$.
Let $\delta = \min(\delta_1, \delta_2)$.
Assume $x \in X$ and $d(x,a) < \delta$.
To show: $|f(x)g(x) - l_1l_2| < \epsilon$.

$$\begin{aligned}
|f(x)g(x) - l_1l_2| &= |(f(x)-l_1)g(x) + l_1(g(x)-l_2)| \\
&\le |(f(x)-l_1)g(x)| + |l_1(g(x)-l_2)|, \quad \text{by the triangle inequality,} \\
&= |(f(x)-l_1)(g(x)-l_2) + (f(x)-l_1)l_2| + |l_1|\,|g(x)-l_2| \\
&\le |f(x)-l_1)(g(x)-l_2)| + |f(x)-l_1)l_2| + |l_1|\,|g(x)-l_2| \\
&\le |f(x)-l_1|\,|g(x)-l_2| + |f(x)-l_1|\,|l_2| + |l_1|\,|g(x)-l_2| \\
&\le \epsilon_1^2 + \epsilon_1|l_2| + |l_1|\epsilon_1 = \epsilon_1(|l_1|+|l_2|+\epsilon_1) \\
&\le \epsilon_1(|l_1|+|l_2|+1) \le \epsilon.
\end{aligned}$$

So $\lim_{x\to a}(f(x)g(x)) = l_1l_2$.

□

Theorem 7.9. (Limits and composition of functions)
Let (X,d) and (Y,ρ) and (Z,σ) be metric spaces.
Let $f\colon Y \to Z$ and $g\colon X \to Y$ be functions and let $a \in X$ and $\ell \in Y$.

Assume that $\lim_{x\to a} g(x)$ *and* $\lim_{x\to a} f(g(x))$ *exist* *and* $\lim_{x\to a} g(x) = \ell$.

Then

$$\lim_{y\to\ell} f(y) = \lim_{x\to a} f(g(x)).$$

Proof.
Let $L = \lim_{y\to\ell} f(y)$.
To show: $\lim_{x\to a} f(g(x)) = L$.
To show: If $\epsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $x \in X$ and $d(x,a) < \delta$ then $\sigma(f(g(x), L) < \epsilon$.
Assume $\epsilon \in \mathbb{R}_{>0}$.
To show: There exists $\delta \in \mathbb{R}_{>0}$ such that if $x \in X$ and $d(x,a) < \delta$ then $\sigma(f(g(x)), L) < \epsilon$.
Since $\lim_{y\to\ell} f(y) = L$, there exists $\delta_1 \in \mathbb{R}_{>0}$ such that if $y \in Y$ and $\rho(y,\ell) < \delta_1$ then $\sigma(f(y), L) < \epsilon$.
Since $\lim_{x\to a} g(x) = \ell$, there exists $\delta \in \mathbb{R}_{>0}$ such that if $x \in X$ and $d(x,a) < \delta$ then $\rho(g(x),\ell) < \delta_1$.
To show: If $x \in X$ and $d(x,a) < \delta$ then $\sigma(f(g(x)), L) < \epsilon$.
Assume $x \in X$ and $d(x,a) < \delta$.

To show: $\sigma(f(g(x)), L) < \epsilon$.
Since $d(x,a) < \delta$ then $\rho(g(x)), \ell) < \delta_1$ and so $\sigma(f(g(x)), L) < \epsilon$.
So $\lim_{x\to a} f(g(x)) = L$.

$\square$

Theorem 7.10. (Limits and order)

(a) Let $(a_1, a_2, \ldots)$ and $(b_1, b_2, \ldots)$ be sequences in $\mathbb{R}$.
Assume that $\lim_{n\to\infty} a_n$ *and* $\lim_{n\to\infty} b_n$ *exist and*

$$\text{if } n \in \mathbb{Z}_{>0} \text{ then } a_n \le b_n. \qquad \text{Then} \qquad \lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n.$$

(b) Let (X,d) be a metric space and let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be functions. Let $a \in X$.
Assume that $\lim_{x\to a} f(x)$ *and* $\lim_{x\to a} g(x)$ *exist and*

$$\text{if } x \in X \text{ then } f(x) \le g(x). \qquad \text{Then} \qquad \lim_{x\to a} f(x) \le \lim_{x\to a} g(x).$$

Proof.

(a) Let $\ell_1 = \lim_{n\to\infty} a_n$ and $\ell_2 = \lim_{n\to\infty} b_n$.
To show: $\ell_1 \le \ell_2$.
Proof by contradiction.
Assume $\ell_1 > \ell_2$.
Let $\epsilon = \ell_1 - \ell_2$.
Let $N_1 \in \mathbb{Z}_{>0}$ be such that if $n \in \mathbb{Z}_{>0}$ and $n > N_1$ then $|a_n - \ell_1| < \frac{\epsilon}{2}$.
Let $N_2 \in \mathbb{Z}_{>0}$ be such that if $n \in \mathbb{Z}_{>0}$ and $n > N_2$ then $|b_n - \ell_2| < \frac{\epsilon}{2}$.
Let $N = \max(N_1, N_2)$.
Then
$$a_N > \ell_1 - \frac{\epsilon}{2} = \ell_1 - \ell_2 + \ell_2 - \frac{\epsilon}{2} = \epsilon + \ell_2 - \frac{\epsilon}{2} = \ell_2 + \frac{\epsilon}{2} > b_N.$$
This is a contradiction to $a_N \le b_N$.
Thus $\lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n$.

(b) Let $\ell_1 = \lim_{x\to a} f(x)$ and $\ell_2 = \lim_{x\to a} g(x)$.
To show: $\ell_1 \le \ell_2$.
Proof by contradiction.
Assume $\ell_1 > \ell_2$.
Let $\epsilon = \ell_1 - \ell_2$.
Let $\delta_1 \in \mathbb{R}_{>0}$ be such that if $x \in X$ and $d(x,a) < \delta_1$ then $|f(x) - \ell_1| < \frac{\epsilon}{2}$.
Let $\delta_2 \in \mathbb{R}_{>0}$ be such that if $x \in X$ and $d(x,a) < \delta_2$ then $|g(x) - \ell_2| < \frac{\epsilon}{2}$.
Let $\delta = \min(\delta_1, \delta_2)$ and let $x \in X$ such that $d(x,a) < \delta$.
Then
$$f(x) > \ell_1 - \frac{\epsilon}{2} = \ell_1 - \ell_2 + \ell_2 - \frac{\epsilon}{2} = \epsilon + \ell_2 - \frac{\epsilon}{2} = \ell_2 + \frac{\epsilon}{2} > g(x).$$
This is a contradiction to $f(x) \le g(x)$.
Thus $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$.

$\square$