10.08.2022

MHS Lect 8 A. Ram ①

Orthogonals and projections

Let $(H, \langle , \rangle)$ be a Hilbert space.

Let W be a closed subspace of H

the <u>orthogonal to W</u> <u>in H</u> is

$$W^\perp = \{x \in H \mid \text{if } w \in W \text{ then } \langle x, w \rangle = 0\}$$

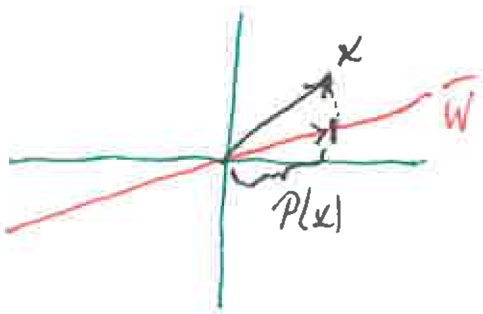
<u>HW</u>: Show that $W \cap W^\perp = 0$.

Projection onto W

A <u>projection onto W</u> is a function $P: H \to H$ such that

(P1) If $x \in H$ then $P(x) \in W$

(P2) If $x \in H$ and $w \in W$ then $\langle x, w \rangle = \langle P(x), w \rangle$.



<u>Proposition</u> Let P and Q be projections onto W. Then

(a) $P = Q$

(b) P is a linear transformation

(c) $P^2 = P$.

(d) $(id - P)^2 = id - P$

Proof (a) To show: If $x \in H$ then $P(x) = Q(x)$.

Assume $x \in H$.

By (P2), if $w \in \overline{W}$ then

$$\langle Q(x), w\rangle = \langle x, w\rangle = \langle P(x), w\rangle.$$

So, if $w \in \overline{W}$ then $0 = \langle P(x), w\rangle - \langle Q(x), w\rangle$

$$= \langle P(x) - Q(x), w\rangle.$$

So $P(x) - Q(x) \in \overline{W}^{\perp}$.

By (P1), $P(x) \in \overline{W}$ and $Q(x) \in \overline{W}$ and since $\overline{W}$ is a subspace $P(x) - Q(x) \in \overline{W}$.

So $P(x) - Q(x) \in \overline{W} \cap \overline{W}^{\perp} = 0$.

So $P(x) - Q(x) = 0$ and $P(x) = Q(x)$.

So $P = Q$.

(b) To show: (ba) If $x_1, x_2 \in H$ then $P(x_1 + x_2) = P(x_1) + P(x_2)$

(bb) If $c \in \mathbb{K}$ and $x \in H$ then $P(cx) = cP(x)$.

(ba) Assume $x_1, x_2 \in H$.

To show: $P(x_1 + x_2) = P(x_1) + P(x_2)$.

By (P2), if $w \in \overline{W}$ then

$$\langle P(x_1 + x_2), w\rangle = \langle x_1 + x_2, w\rangle = \langle x_1, w\rangle + \langle x_2, w\rangle$$

$= \langle P(x_1), w\rangle + \langle P(x_2), w\rangle = \langle P(x_1) + P(x_2), w\rangle$

So $\langle P(x_1+x_2) - (P(x_1)+P(x_2)), w\rangle = 0$ if $w \in \overline{W}$.

So $P(x_1+x_2) - (P(x_1)+P(x_2)) \in \overline{W}^{\perp}$.

By (P1) and the fact that $\overline{W}$ is a subspace,

$P(x_1+x_2) - (P(x_1)+P(x_2)) \in \overline{W}$.

So $P(x_1+x_2) - (P(x_1)+P(x_2)) \in \overline{W} \cap \overline{W}^{\perp} = 0$.

So $P(x_1+x_2) - (P(x_1)+P(x_2)) = 0$.

HW Do a similar proof for (bb) using

$\langle P(cx), w\rangle = \langle cx, w\rangle = c\langle x, w\rangle = c\langle P(x), w\rangle$

$= \langle cP(x), w\rangle$ and $\overline{W} \cap \overline{W}^{\perp} = 0$

HW Do a similar proof for (c) using

$\langle P(x), w\rangle = \langle P(P(x)), w\rangle = \langle P^2(x), w\rangle$

and $\overline{W} \cap \overline{W}^{\perp} = 0$.

Then $(id-P)^2 = id - 2P + P^2 = id - 2P + P = id - P$. //

Corollary. $H = \overline{W} \oplus \overline{W}^{\perp}$.

Proof To show: (a) $W \cap \overline{W}^{\perp}$

(b) $H = \overline{W} + \overline{W}^{\perp}$

(b) To show: If $x \in H$ then $x \in \overline{W} + \overline{W}^{\perp}$

Assume $x \in H$.

Then $x = P(x) + (id-P)(x)$ and $P(x) \in \overline{W}$ and $(id-P)(x) \in \overline{W}^{\perp}$

Do projections exist?

Define, for $x \in H$,

$$d(x, \overline{W}) = \inf\{ d(x,w) \mid w \in \overline{W} \}$$

Then define $P_{\overline{W}} : H \to H$ by $P_{\overline{W}}(x) = y$,
where

$$y \in \overline{W} \text{ such that } d(x,y) = d(x, \overline{W}).$$

Does y exist?

Theorem Yes and $P_{\overline{W}}$ is a projection onto $\overline{W}$

Another construction of a projection

Assume $(e_1, e_2, \ldots)$ is an orthonormal sequence in H and let

$$\overline{W} = \overline{\mathrm{span}\{e_1, e_2, \ldots\}}$$

Define $P : H \to H$ by

$$P(x) = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n, \quad \text{for } x \in H.$$

Proposition $P : H \to H$ is a projection onto $\overline{W}$ and

$$H = \overline{W} \oplus \overline{W}^{\perp}.$$