The mean value theorem

<u>Theorem</u> Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let $f: X \to Y$ be a continuous function. Let $E \subseteq X$.

(a) If E is connected then $f(E)$ is connected.

(b) If E is compact then $f(E)$ is compact.

<u>Proof</u> To show: If $f(E)$ is not connected then E is not connected

Assume $f(E)$ is not connected.

Let A and B be open in $f(E)$ such that

$A \neq \emptyset$ and $B \neq \emptyset$ and $A \cup B \supseteq f(E)$ and $A \cap B = \emptyset$

Let $C = f^{-1}(A)$ and $D = f^{-1}(B)$.

Then

$$C \cup D = f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) \supseteq f^{-1}(f(E)) \supseteq E.$$

$$C \cap D = f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(\emptyset) = \emptyset.$$

Also

$C = f^{-1}(A) \neq \emptyset$ since $A \neq \emptyset$ and $A \subseteq f(E)$

$D = f^{-1}(B) \neq \emptyset$ since $B \neq \emptyset$ and $B \subseteq f(E)$

So E is not connected.

Theorem Let $E \subseteq \mathbb{R}$ (with the standard topology).
Then
(a) E is connected if and only if E is an interval.

(b) E is compact if and only if E is closed and bounded.

Corollary Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function. If $a, b \in \mathbb{R}$ with $a < b$ then there exist m and M in $\mathbb{R}$ with $m < M$ and

$$f([a,b]) = [m, M].$$

This is the intermediate value theorem (strong form).

Theorem (the mean value theorem)
Let $a, b \in \mathbb{R}$ with $a < b$. If $f: [a,b] \to \mathbb{R}$ is continuous and the derivative $f': (a,b) \to \mathbb{R}$ exists then there exists

$c \in (a,b)$ such that $f(b) = f(a) + f'(c)(b-a)$.

Proof To show: There exists $c \in \mathbb{R}_{(a,b)}$ such that

$$f(b) = f(a) + f'(c)(b-a).$$

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<u>Case 1</u>: $f(a) = f(b)$

By the intermediate value theorem there exist $\min, \max \in \mathbb{R}$ such that

$$f(\mathbb{R}_{[a,b]}) = \mathbb{R}_{[\min, \max]}.$$

So there exists $c \in \mathbb{R}_{(\max, \cancel{\max})}$ such that

$$f(c) = \max.$$

Thus, if $\varepsilon \in \mathbb{R}_{>0}$ such that $c-\varepsilon \in (a,b)$ and $c+\varepsilon \in \mathbb{R}_{(a,b)}$ then

$$f(c+\varepsilon) \le f(c) \text{ and } f(c-\varepsilon) \le f(c)$$

So $f'(c) \ge 0$ and $f'(c) \le 0$.

So $f'(c) = 0$ and

$$f(b) = f(a) + 0 = f(a) + f'(c)(b-a).$$

<u>Case 2</u>: $f(a) \ne f(b)$

Let $g(x) = -\left(\dfrac{f(b)-f(a)}{b-a}\right)(x-a) + f(x)$

Then $g(a) = f(a)$ and $g(b) = f(a)$ and

$$g'(x) = -\left(\frac{f(b)-f(a)}{b-a}\right) + f'(x)$$

By case 1 there exists $c \in \mathbb{R}_{(a,b)}$ with $g'(c) = 0$

So $f'(c) - \dfrac{f(b)-f(a)}{b-a} = 0.$

So $f(b) = f(a) + f'(c)(b-a).$ //

<u>Theorem</u> (Taylor's theorem)
Assume $a, b \in \mathbb{R}$ with $a < b$. If $f : \mathbb{R}_{[a,b]} \to \mathbb{R}$ is such that $f' : \mathbb{R}_{[a,b]} \to \mathbb{R}, \ldots, f^{(r)} : \mathbb{R}_{(a,b)} \to \mathbb{R}$ exist then there exists $c \in \mathbb{R}_{(a,b)}$ such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{1}{2!} f''(a)(b-a)^2 + \cdots$$

$$\cdots + \frac{1}{(r-1)!} f^{(r-1)}(a)(b-a)^{r-1} + \frac{1}{r!} f^{(r)}(c)(b-a)^r.$$