Connected sets

Let $(X, \mathcal{T}_X)$ be a topological space.
The space X is *connected* if there do not exist

$$U \in \mathcal{T}_X,\ V \in \mathcal{T}_X \text{ with } U \cup V = X \text{ and } U \cap V = \emptyset,\ U \neq \emptyset \text{ and } V \neq \emptyset.$$

Let $A \subseteq X$. The *subspace topology on A* is

$$\mathcal{T}_A = \{ U \cap A \mid U \in \mathcal{T}_X \}$$

The set A is *connected in X* if $(A, \mathcal{T}_A)$ is connected.

Let $(S, \leq)$ be a poset.
Let $E \subseteq S$.
The set E is an *interval* if E satisfies
if $x, y \in E$ and $z \in S$ and $x < z < y$
then $z \in E$

Theorem Let $\mathbb{R}$ have the standard topology.
Let $E \subseteq \mathbb{R}$. Then
E is connected if and only if
E is an interval
(with respect to the standard ordering on $\mathbb{R}$
($x \leq y$ if there exists $z \in \mathbb{R}_{\geq 0}$ with $x + z = y$).

Proof

$\Rightarrow$ Assume E is not an interval.

Let $x, y \in E$ and $z \in E^c$ with $x < z < y$.

Let $A = \mathbb{R}_{<z} \cap E$ and $B = \mathbb{R}_{>z} \cap E$.

Then A and B are open sets of E and since $x \in A$ and $y \in B$ then

$$A \neq \emptyset,\ B \neq \emptyset,\ A \cap B = \emptyset \text{ and } A \cup B = E.$$

So E is not connected.

$\Leftarrow$ Assume E is an interval.

To show: E is connected.

Let $A \subseteq E$ and $B \subseteq E$ be open sets of E such that

$$A \neq \emptyset,\ B \neq \emptyset \text{ and } A \cup B = E.$$

To show: $A \cap B \neq \emptyset$.

To show: There exists $z \in A \cap B$.

Let $x_1, y_1 \in E$ with $x_1 \in A$ and $y_1 \in B$.

Switching A and B if necessary assume $x_1 < y_1$

Construct sequences $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ by

$$x_{i+1} = \tfrac{1}{2}(x_i + y_i) \text{ and } y_{i+1} = y_i, \quad \text{if } \tfrac{1}{2}(x_i + y_i) \in A$$

$$x_{i+1} = x_i \text{ and } y_{i+1} = \tfrac{1}{2}(x_i + y_i) \quad \text{if } \tfrac{1}{2}(x_i + y_i) \in B.$$

By induction $x_i \in E$ and $y_i \in E$ and since E is an interval then $\frac{1}{2}(x_i + y_i) \in E$.

So
$$x_{i+1} \in E \text{ and } y_{i+1} \in E.$$

Also, by construction
$$x_{i+1} \in A \text{ and } y_{i+1} \in B. \text{ and } x_i \le x_{i+1} < y_{i+1} \le y_i.$$

So
$$|x_{i+1} - y_{i+1}| \le \frac{1}{2}|x_i - y_i| \text{ so that}$$
$$|x_{i+1} - y_{i+1}| \le \frac{1}{2^k}|x_1 - y_1|.$$

Since increasing bounded sequences converge in $\mathbb{R}$ and $(x_1, x_2, \ldots)$ is increasing and bounded by y_1

then $\lim_{n\to\infty} x_n$ exists in $\mathbb{R}$.

Since decreasing bounded sequences converge in $\mathbb{R}$ and $(y_1, y_2, \ldots)$ is decreasing and bounded by x_1

then $\lim_{n\to\infty} y_n$ exists in $\mathbb{R}$.

Since
$$\lim_{n\to\infty} |x_n - y_n| = 0 \text{ then } \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n.$$

Let
$$z = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n.$$

Since
$$x_1 \le x_2 \le \cdots \le x_n < y_n \le y_{n-1} \le \cdots \le y_1 \quad \text{for } n \in \mathbb{Z}_{>0}$$
then
$$x_1 \le z \le y_1$$
Since z is an interval then $z \in E$.
Then
$$z = \lim_{n\to\infty} x_n \in \overline{A} \quad \text{and} \quad z = \lim_{n\to\infty} y_n \in \overline{B}.$$

__Case 1:__ If $z \in A$ then there exists $\varepsilon \in \mathbb{E}$ with
$$B_\varepsilon(z) \subseteq A \quad (\text{since } A \text{ is open})$$
and $B_\varepsilon(z) \cap B \neq \phi$ (since $z \in \overline{B}$)

So $A \cap B \supseteq B_\varepsilon(z) \cap B \neq \phi$.

__Case 2:__ If $z \in B$ then there exists $\varepsilon \in \mathbb{E}$ with
$$B_\varepsilon(z) \subseteq B \quad (\text{since } B \text{ is open})$$
and $B_\varepsilon(z) \cap A \neq \phi$ (since $z \in \overline{A}$).

So $A \cap B \supseteq B_\varepsilon(z) \cap A \neq \phi$.

So $A \cap B \neq \phi$.

So E is connected //