Uniform spaces

Let X be a set. For $E \subseteq X \times X$ let

$$\sigma(E) = \{(y,x) \mid (x,y) \in E\}$$

$$E \times_X E = \left\{ (x,y) \;\middle|\; \begin{array}{l} \text{there exists } z \in X \text{ with} \\ (x,z) \in E \text{ and } (z,y) \in E \end{array} \right\}$$

and let

$$\Delta(X) = \{(x,x) \mid x \in X\}.$$

A uniform space is a set X with a collection $\mathcal{E}_X$ of subsets of $X \times X$ such that

(a) If $E \in \mathcal{E}_X$ then $\Delta(X) \subseteq E$

(b) If $E \in \mathcal{E}_X$ and $D \subseteq X \times X$ and $D \supseteq E$ then $D \in \mathcal{E}_X$

(c) If $\ell \in \mathbb{Z}_{>0}$ and $E_1, E_2, \ldots, E_\ell \in \mathcal{E}_X$ then
$$E_1 \cap E_2 \cap \cdots \cap E_\ell \in \mathcal{E}_X$$

(d) If $E \in \mathcal{E}_X$ then $\sigma(E) \in \mathcal{E}_X$

(e) If $E \in \mathcal{E}_X$ then there exists
$$D \in \mathcal{E}_X \text{ such that } D \times_X D \subseteq E.$$

Let $(X, \mathcal{E}_X)$ be a uniform space

A fatdiagonal, or entourage, is a set $E \in \mathcal{E}_X$.

Neighborhoods and open sets

Let $(X, \mathcal{E}_X)$ be a uniform space.

For $x \in X$ and $E \in \mathcal{E}_X$ the <u>E-neighborhood at x</u>

is $B_E(x) = \{ y \in X \mid (x,y) \in E \}$.

<u>Neighborhoods of x</u> (uniform spaces).

$$\mathcal{N}(x) = \left\{ N \subseteq X \;\middle|\; \begin{array}{c} \text{there exists } E \in \mathcal{E}_X \\ \text{with } N \supseteq B_E(x) \end{array} \right\}$$

<u>Neighborhoods of x</u> (metric spaces).

$$\mathcal{N}(x) = \left\{ N \subseteq X \;\middle|\; \begin{array}{c} \text{there exists } \varepsilon \in \mathbb{E} \\ \text{with } N \supseteq B_\varepsilon(x) \end{array} \right\}$$

<u>Open sets</u> (uniform spaces)

$$\mathcal{T}_X = \left\{ U \subseteq X \;\middle|\; \begin{array}{c} \text{if } x \in U \text{ then there exists} \\ E \in \mathcal{E}_X \text{ with } B_E(x) \subseteq U \end{array} \right\}$$

This is the <u>uniform space topology</u> on $(X, \mathcal{E}_X)$.

<u>Open sets (metric spaces)</u>

$$\mathcal{T}_X = \left\{ U \subseteq X \;\middle|\; \begin{array}{c} \text{if } x \in U \text{ then there exists} \\ \varepsilon \in \mathbb{E} \text{ with } B_\varepsilon(x) \subseteq U \end{array} \right\}$$

This is the <u>metric space topology</u>.

12.09.2022 (3)
MHS Lect 22.

Uniformly continuous functions

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.
Let $f: X \to Y$ be a function. The function f is <u>continuous</u> if f satisfies:

$$\text{if } V \in \mathcal{T}_Y \text{ then } f^{-1}(V) \in \mathcal{T}_X$$

Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be uniform spaces.
Let $f: X \to Y$ be a function. The function f is <u>uniformly continuous</u> if f satisfies:

$$\text{if } E \in \mathcal{E}_Y \text{ then } (f \times f)^{-1}(E) \in \mathcal{E}_X.$$

<u>Proposition</u> Let (X, d_X) and (Y, d_Y) be metric spaces.
Let $f: X \to Y$ be a function.

(a) The function f is continuous if and only if f satisfies:

if $\varepsilon \in \mathbb{E}$ and $a \in X$ then there exists $\delta \in \mathbb{E}$ such that

if $x \in X$ and $d_X(a,x) < \delta$ then $d_Y(f(a), f(x)) < \varepsilon$.

(b) The function f is uniformly continuous if and only if f satisfies:

if $\varepsilon \in \mathbb{E}$ then there exists $\delta \in \mathbb{E}$ such that

if $x \in X$ and $a \in X$ and $d_X(a,x) < \delta$ then $d_Y(f(a), f(x)) < \varepsilon$

<u>Proposition</u> Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be uniform spaces. Let $f: X \to Y$ be a function.

(a) The function f is continuous if and only if f satisfies

if $E \in \mathcal{E}_Y$ and $a \in X$ then there exists $D \in \mathcal{E}_X$ such that

if $x \in X$ and $(a, x) \in D$ then $(f(a), f(x)) \in E$.

(b) The function f is uniformly continuous if and only if f satisfies:

if $E \in \mathcal{E}_Y$ then there exists $D \in \mathcal{E}_X$ such that

if $x \in X$ and $a \in X$ and $(a, x) \in D$ then $(f(a), f(x)) \in E$.

<u>Proposition</u> Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be uniform spaces. Let $\mathcal{T}_X$ be the uniform space topology on $(X, \mathcal{E}_X)$. Let $\mathcal{T}_Y$ be the uniform space topology on $(Y, \mathcal{E}_Y)$. Let $f: X \to Y$ be a function.

If f is uniformly continuous then f is continuous.