

New topological spaces from old

Subspaces Let (X, \mathcal{T}_X) be a topological space.

Let S be a subset of X . Then

$$\mathcal{T}_S = \{S \cap U \mid U \in \mathcal{T}_X\}$$

is a topology on S , the subspace topology.

The subspace topology is the minimal topology on S such that

$$\begin{array}{ccc} S & \longrightarrow & X \\ s & \longmapsto & s \end{array} \text{ is a continuous function.}$$

The quotient topology

Let (X, \mathcal{T}_X) be a topological space.

Let \sim be an equivalence relation on X .

An equivalence relation on X is a subset R of $X \times X$,

$$R = \{(s, t) \mid s \sim t\},$$

satisfying

(a) if $s \in X$ then $s \sim s$,

(b) if $s, t \in X$ and $s \sim t$ then $t \sim s$.

(c) if $s, t, u \in X$ and $s \sim t$ and $t \sim u$ then $s \sim u$.

Let $s \in X$. The equivalence class of s is

$$[s] = \{t \in X \mid t \sim s\}.$$

The quotient of X by \sim is the set

$$X/\sim = \{[s] \mid s \in X\}.$$

The quotient topology is the minimal topology on X/\sim such that

$X \rightarrow X/\sim$
 $s \mapsto [s]$ is a continuous function.

Products Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces. The product topology is the topology on $X \times Y$ generated by

$$\mathcal{B} = \{U \times V \mid U \in \mathcal{T}_X \text{ and } V \in \mathcal{T}_Y\}.$$

The product topology is the minimal topology on $X \times Y$ such that

$$\begin{aligned} X \times Y &\rightarrow X & \text{and} & \quad X \times Y \rightarrow Y \\ (x, y) &\mapsto x & & (x, y) \mapsto y \end{aligned}$$

are continuous functions.

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MHS Lect 18

Universal property Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces. The product of X and Y , is

(a) a topological space $(X \times Y, \mathcal{T}_{X \times Y})$ with continuous functions

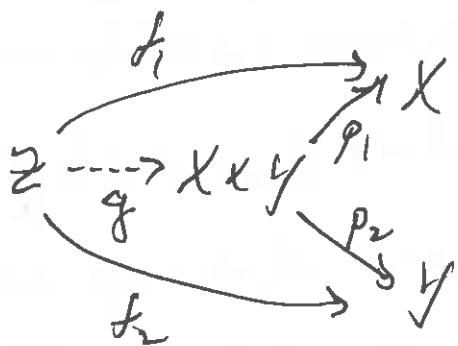
$$X \times Y \xrightarrow{P_1} X \text{ and } X \times Y \xrightarrow{P_2} Y$$

(b) If (Z, \mathcal{T}_Z) is a topological space with continuous functions

$$Z \xrightarrow{f_1} X \text{ and } Z \xrightarrow{f_2} Y$$

then there exists a unique continuous function

$$g: Z \rightarrow X \times Y \text{ such that } f_1 = P_1 \circ g \quad f_2 = P_2 \circ g.$$



Continuous Functions

Continuous functions are for comparing topological spaces

Categories

Categories have Objects and Morphisms

Morphisms are for comparing objects.

Examples:

<u>Category</u>	<u>Objects</u>	<u>Morphisms</u>
Sets	X	functions
Topological spaces	(X, \mathcal{T}_X)	continuous functions
Groups	(G, \cdot)	homomorphisms
Vector spaces	$(V, \begin{matrix} \text{addition} \\ \text{and} \\ \text{scalar mult} \end{matrix})$	linear transformations

Point of "category theory": (universal property)

If you know the definition of product for one category then you know it for all categories.

Products of metric spaces?

Let (X, d_X) and (Y, d_Y) be metric spaces.

Some favourite metrics on $X \times Y$ are

(1) $d_1 : (X \times Y) \times (X \times Y) \rightarrow \mathbb{R}_{\geq 0}$ given by

$$d_1((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2).$$

(2) $d_2 : (X \times Y) \times (X \times Y) \rightarrow \mathbb{R}_{\geq 0}$ given by

$$d_2((x_1, y_1), (x_2, y_2)) = \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}$$

(3) $d_\infty : (X \times Y) \times (X \times Y) \rightarrow \mathbb{R}_{\geq 0}$ given by

$$d_\infty((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$$

(4) Let $p \in \mathbb{R}_{>1}$. Let $d_p : (X \times Y) \times (X \times Y) \rightarrow \mathbb{R}_{\geq 0}$ given by

$$d_p((x_1, y_1), (x_2, y_2)) = \left(d_X(x_1, x_2)^p + d_Y(y_1, y_2)^p \right)^{\frac{1}{p}}.$$

Any of these might be called the product metric spaces.

Let

\mathcal{T}_1 be the metric space topology for $(X \times Y, d_1)$

\mathcal{T}_2 be the metric space topology for $(X \times Y, d_2)$

\mathcal{T}_{∞} be the metric space topology for $(X \times Y, d_{\infty})$

\mathcal{T}_p be the metric space topology for $(X \times Y, d_p)$

Let

\mathcal{T}_X be the metric space topology for (X, d_X)

\mathcal{T}_Y be the metric space topology for (Y, d_Y) .

Let

$\mathcal{T}_{X \times Y}$ be the product topology on $X \times Y$ coming from (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) .

Show that

$$\mathcal{T}_{X \times Y} = \mathcal{T}_1 \times \mathcal{T}_2 = \mathcal{T}_{\infty} = \mathcal{T}_p.$$