Limit points, cluster points and the metric space topology

Let $(X, \mathcal{T}_X)$ be a topological space.

Let $A \subseteq X$. A close point to A is $z \in X$ such that

if $N \in \mathcal{N}(z)$ then $N \cap A \neq \emptyset$.

Let $(a_1, a_2, \ldots)$ be a sequence in A.

A limit point of $(a_1, a_2, \ldots)$ is $z \in X$ such that

if $N \in \mathcal{N}(z)$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that $N \supseteq \{a_\ell, a_{\ell+1}, \ldots\}$.

i.e. $\lim_{n \to \infty} a_n = z$.

A cluster point of $(a_1, a_2, \ldots)$ is $z \in X$ such that there exists a subsequence $(a_{n_1}, a_{n_2}, \ldots)$ with $\lim_{k \to \infty} a_{n_k} = z$.

<u>Example</u> Topologies on a set with two points $X = \{p_1, p_2\}$.

$$\left\{\begin{matrix} X \\ \{p_1\} \ \{p_2\} \\ \phi \end{matrix}\right\}, \quad \left\{\begin{matrix} X \\ \{p_1\} \\ \phi \end{matrix}\right\}, \quad \left\{\begin{matrix} X \\ \{p_2\} \\ \phi \end{matrix}\right\}, \quad \left\{\begin{matrix} X \\ \\ \phi \end{matrix}\right\}$$

<u>discrete topology</u>
all subsets are open

<u>trivial topology</u>
only ϕ and X are open

Consider $X = \{p_1, p_2\}$ with $\mathcal{T}_X = \left\{\begin{matrix} X \\ \{p_1\} \\ \phi \end{matrix}\right\}$

Let $A = \{p_1\}$.

Then $A^\circ = \{p_1\}$ and $\bar{A} = X = \{p_1, p_2\}$.

Let $(a_1, a_2, \ldots)$ be the sequence $(p_1, p_1, \ldots)$.

Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} p_1 = \{p_1, p_2\}.$$

Since $N(p_2) = \{X\}$ (X is the only neighborhood of p_2)

So limit points of sequences are not unique.

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The metric space topology

Let (X, d_X) be a metric space.
Let

$$\mathcal{T}_X = \left\{ U \subseteq X \ \middle|\ \begin{array}{l} \text{if } a \in U \text{ then there exists} \\ \varepsilon \in \mathbb{E} \text{ such that } B_\varepsilon(a) \subseteq U \end{array} \right\}$$

(a) $\mathcal{T}_X$ is a topology on X.

(b) If $A \subseteq X$ then

$$\overline{A} = \left\{ z \in X \ \middle|\ \begin{array}{l} \text{there exists a sequence} \\ (a_1, a_2, \ldots) \text{ in } A \text{ with } \lim_{n\to\infty} a_n = z \end{array} \right\}$$

(c) If $(a_1, a_2, \ldots)$ is a sequence in A then $(a_1, a_2, \ldots)$ has at most one limit point in X.

For the metric space topology,

$$N(z) = \left\{ N \subseteq X \ \middle|\ \begin{array}{l} \text{there exists } \varepsilon \in \mathbb{E} \\ \text{with } B_\varepsilon(z) \subseteq N \end{array} \right\}$$

__Proof of (c)__ Assume $(a_1, a_2, \ldots)$ is a sequence in A and there exist $z_1, z_2 \in X$ such that

$$\lim_{n\to\infty} a_n = z_1 \quad \text{and} \quad \lim_{n\to\infty} a_n = z_2.$$

Let $\varepsilon = d_X(z_1, z_2)$. Let

$$N_1 = B_{\le \varepsilon/3}(z_1) = \{x \in X \mid d_X(x, z_1) \le \varepsilon/3\}$$

$$N_2 = B_{\le \varepsilon/3}(z_2) = \{x \in X \mid d_X(x, z_2) \le \varepsilon/3\}.$$

Let $\ell_1 \in \mathbb{Z}_{>0}$ such that $B_{\le \varepsilon/3}(z_1) \supseteq \{a_{\ell_1}, a_{\ell_1+1}, \ldots\}$.

$\ell_2 \in \mathbb{Z}_{>0}$ such that $B_{\le \varepsilon/3}(z_2) \supseteq \{a_{\ell_2}, a_{\ell_2+1}, \ldots\}$.

Let $\ell = \max\{\ell_1, \ell_2\}$. Then

$$a_\ell \in B_{\le \varepsilon/3}(z_1) \cap B_{\le \varepsilon/3}(z_2).$$

By the triangle inequality,

$$\varepsilon = d_X(z_1, z_2) \le d_X(z_1, a_\ell) + d_X(a_\ell, z_2) \le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}$$

So $\varepsilon = 0$ and $z_1 = z_2$.

<u>Proof of (b)</u> Let

$$R = \left\{ z \in X \;\middle|\; \begin{matrix} \text{there exists a sequence} \\ (a_1, a_2, \ldots) \text{ in } A \text{ with } \lim_{n\to\infty} a_n = z \end{matrix} \right\}$$

$$\overline{A} = \{z \in X \mid z \text{ is a close point to } A\}.$$

To show: (ba) $\overline{A} \subseteq R$

(bb) $R \subseteq \overline{A}$.

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(ba) Let $z \in \overline{A}$

To show: $z \in R$.

To show: There exists a sequence $(a_1, a_2, \ldots)$ in A with $\lim_{n\to\infty} a_n = z$.

Let

$$a_1 \in B_{0.1}(z) \cap A, \quad a_2 \in B_{0.01}(z) \cap A, \ \ldots$$

where we are using that z is a close point to A.

To show: $\lim_{n\to\infty} a_n = z$.

To show: If $N \in \mathcal{N}(z)$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that $N \supseteq \{a_\ell, a_{\ell+1}, \ldots\}$.

Let $N \in \mathcal{N}(z)$

To show: There exists $\ell \in \mathbb{Z}_{>0}$ such that

$$N \supseteq \{a_\ell, a_{\ell+1}, \ldots\}.$$

Let $\ell \in \mathbb{Z}_{>0}$ such that $B_{10^{-\ell}}(z) \subseteq N$.

Then

$$\{a_\ell, a_{\ell+1}, \ldots\} \subseteq B_{10^{-\ell}}(z) \subseteq N.$$

So $\lim_{n\to\infty} a_n = z$.

(bb) Let $z \in R$

Let $(a_1, a_2, \ldots)$ be a sequence in A with $\lim_{n \to \infty} a_n = z$.

To show: $z \in \overline{A}$.

To show: If $N \in \mathcal{N}(z)$ then $N \cap A \neq \emptyset$.

Assume $N \in \mathcal{N}(z)$.

Since $\lim_{n \to \infty} a_n = z$ then there exists $\ell \in \mathbb{Z}_{>0}$ with $\{a_\ell, a_{\ell+1}, \ldots\} \subseteq N$.

So $N \cap A \neq \emptyset$.

So $z \in \overline{A}$. and $R \subseteq \overline{A}$.

So $R = \overline{A}$. //.