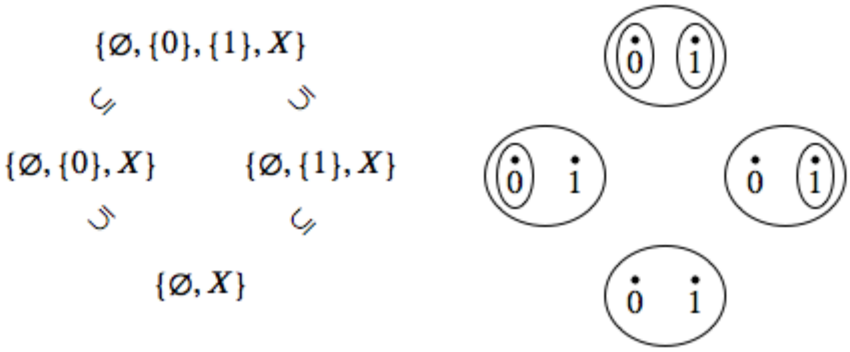
(c) Finite intersections of open sets in X are open in X.

In other words, a *topology* on X is a set $\mathcal{T}$ of subsets of X such that

(a) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$,
(b) If $\mathcal{S} \subseteq \mathcal{T}$ then $\left(\bigcup_{U\in\mathcal{S}} U\right) \in \mathcal{T}$,
(c) If $\ell \in \mathbb{Z}_{>0}$ and $U_1, U_2, \ldots, U_\ell \in \mathcal{T}$ then $U_1 \cap U_2 \cap \cdots \cap U_\ell \in \mathcal{T}$.

A *topological space* is a set X with a topology $\mathcal{T}$ on X. An *open set in* X is a set in $\mathcal{T}$.

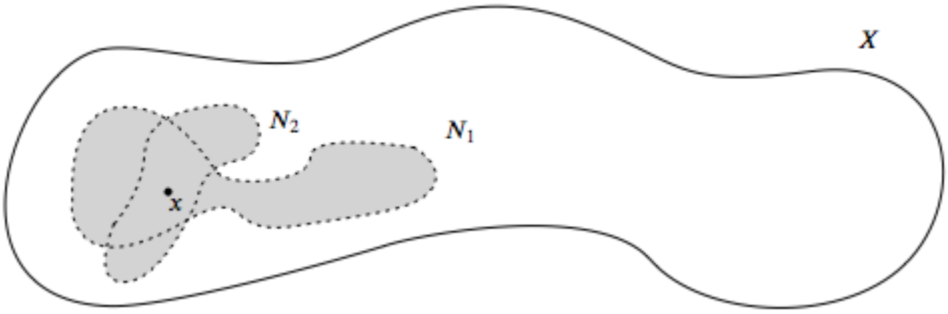


The four possible topologies on $X = \{0, 1\}$.

In a topological space, perhaps even more important than the open sets are the neighborhoods. Let $(X, \mathcal{T})$ be a topological space. Let $x \in X$. The *neighborhood filter of* x is

$$\mathcal{N}(x) = \{N \subseteq X \mid \text{there exists } U \in \mathcal{T} \text{ such that } x \in U \text{ and } N \supseteq U\}.$$

A *neighborhood of* x is a set in $\mathcal{N}(x)$.



Neighborhoods of x.

Let $(X, \mathcal{T})$ be a topological space.

A *closed set in* X is $K \subseteq X$ such that the complement $X - K$ is open.

Let $A \subseteq X$. A *close point* to A is an element $x \in X$ such that

$$\text{if } N \in \mathcal{N}(x) \text{ then } N \cap A \neq \emptyset.$$

The *closure of* A is the subset $\overline{A}$ of X such that

(a) $\overline{A}$ is closed in X and $\overline{A} \supseteq A$,
(b) If C is closed in X and $C \supseteq A$ then $C \supseteq \overline{A}$.

Proposition 4.6. *Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. The closure of A is the set of close points of A.*

4.3 Filters

Let X be a set. A *filter on* X is a collection $\mathcal{F}$ of subsets of X such that

(a) $\emptyset \notin \mathcal{F}$.
(b) (upper ideal) If $N \in \mathcal{F}$ and E is a subset of X with $N \subseteq E$ then $E \in \mathcal{F}$,
(c) (closed under finite intersection) If $\ell \in \mathbb{Z}_{>0}$ and

$$N_1, N_2, \ldots, N_\ell \in \mathcal{F} \qquad \text{then} \qquad N_1 \cap N_2 \cap \cdots \cap N_\ell \in \mathcal{F},$$

An *ultrafilter on* X is a maximal filter on X, i.e. an ultrafilter on X is a filter $\mathcal{G}$ on X such that

$$\text{if } \mathcal{F} \text{ is a filter on } X \text{ and } \mathcal{F} \supseteq \mathcal{G} \text{ then } \mathcal{F} = \mathcal{G}.$$

Let $(X, \mathcal{T})$ be a topological space and let $z \in Z$. The *neighborhood filter of* z is

$$\mathcal{N}(z) = \{N \subseteq X \mid \text{there exists } U \in \mathcal{T} \text{ such that } x \in U \text{ and } N \supseteq U\}$$

Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{F}$ be a filter on X.

A *limit point of* $\mathcal{F}$ is $z \in X$ such that $\mathcal{F} \supseteq \mathcal{N}(z)$.

A *cluster point of* $\mathcal{F}$ is $z \in X$ such that there exists a filter $\mathcal{G}$ on X with $\mathcal{G} \supseteq \mathcal{F}$ and z is a limit point of $\mathcal{G}$.

4.4 Hausdorff topological spaces

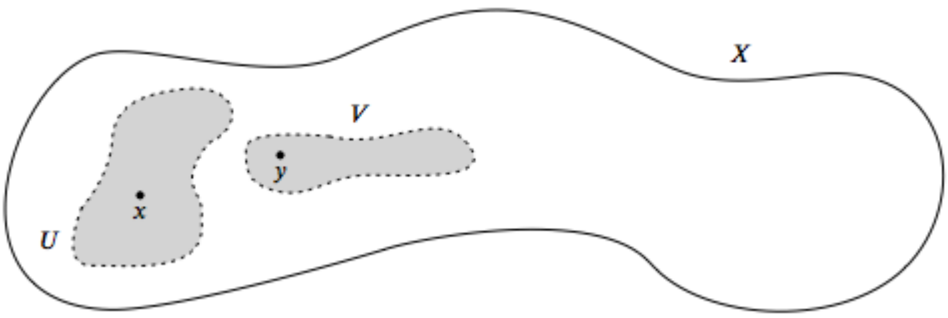
The goal of this section is to explain that if $(X, \mathcal{T})$ is a topological space then

$$\text{limit unique} \quad \Leftrightarrow \quad \text{Hausdorff} \quad \Leftrightarrow \quad \text{separated} \quad \Leftrightarrow \quad \text{neighborhood pinpointed} \qquad \text{(H)}$$

The definitions of these terms are as follows. Let $(X, \mathcal{T})$ be a topological space.

- The space $(X, \mathcal{T})$ is *limit unique* if every filter on X has at most one limit point.
- The space $(X, \mathcal{T})$ is *Hausdorff* if $(X, \mathcal{T})$ satisfies

  if $x, y \in X$ and $x \neq y$ then there exists $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$ such that $U \cap V = \emptyset$,



The Hausdorff property

- The space $(X,\mathcal{T})$ is *separated* if
$$\Delta(X) = \{(x,x) \mid x \in X\} \text{ is a closed subset of } X \times X$$
(with the product topology on $X \times X$).

- The space $(X,\mathcal{T})$ is *neighborhood pinpointed* if $(X,\mathcal{T})$ satisfies
$$\text{if } x \in X \quad \text{then} \quad \bigcap_{N\in\mathcal{N}(x)} \overline{N} = \{x\}.$$

4.4.1 Sketch of the proof of the equivalences in (H)

Theorem 4.7. *The following conditions on a topological space $(X,\mathcal{T})$ are equivalent.*

(limit unique) If $\mathcal{G}$ is a filter on X then $\mathcal{G}$ has at most one limit point.

(cluster unique) If $\mathcal{F}$ is a filter on X and x is a limit point of $\mathcal{F}$ then x is the only cluster piont of $\mathcal{F}$.

(neighborhood pinpointed) If $x \in X$ then $\displaystyle\bigcap_{N\in\mathcal{N}(x)} \overline{N} = \{x\}$.

(Hausdorff) If $x, y \in X$ and $x \neq y$ then there exist $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$ such that $U \cap V = \emptyset$.

(separated) $\Delta(X) = \{(x,x) \mid x \in X\}$ is a closed subset of $X \times X$ (with the product topology on $X \times X$).

Sketch of proof.

Hausdorff $\Leftrightarrow$ separated: The point here is that if $x, y \in X$ with $x \neq y$ then $(x,y) \in X \times X$ is not a close point to $\Delta(X)$ if and only if there exist $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$ such that $(U \times V) \cap \Delta(X) = \emptyset$ and this happens if and only if there exists $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$ such that $U \cap V = \emptyset$.

Hausdorff $\Leftrightarrow$ neighborhood pinpointed $\Leftrightarrow$ (H6) $\Leftrightarrow$ limit unique: The point here is that if $(X,\mathcal{T})$ is Hausdroff holds and $x, y \in X$ with $y \neq x$ then $y \notin \bigcap_{U\in\mathcal{N}(x)} \overline{U}$ which is equivalent to $\{x\} = \bigcap_{U\in\mathcal{N}(x)} \overline{U}$ so that x is the only cluster point of $\mathcal{N}(x)$. If $\mathcal{F}$ is a filter with x as a limit point then x is also a cluster point of $\mathcal{F}$ and
$$x \in \bigcap_{M\in\mathcal{F}} \overline{M} \subseteq \bigcap_{U\in\mathcal{N}(x)} \overline{U} = \{x\}.$$
□

4.5 Compact topological spaces

The goal of this section is to explain that if $(X,\mathcal{T})$ is a topological space then
$$\text{filter compact} \Leftrightarrow \text{ultrafilter compact} \Leftrightarrow \text{exclusion compact} \Leftrightarrow \text{cover compact} \qquad \text{(C)}$$
The definitions of these terms are as follows. Let $(X,\mathcal{T})$ be a topological space.

- The space $(X,\mathcal{T})$ is *filter compact* if every filter has a cluster point.
- The space $(X,\mathcal{T})$ is *ultrafilter compact* if every ultrafilter has a limit point.
- The space $(X,\mathcal{T})$ is *exclusion compact* if every closed exclusion contains a finite exclusion, i.e.

  If $\mathcal{C}$ is a collection of closed sets of X such that $\displaystyle\bigcap_{K\in\mathcal{C}} K = \emptyset$

  then there exists $\ell \in \mathbb{Z}_{>0}$ and $K_1, K_2, \ldots, K_\ell \in \mathcal{C}$ such that $K_1 \cap K_2 \cap \cdots \cap K_\ell = \emptyset$.

4.6.1 Spaces

Although it is traditional to define topological spaces via axioms for **open sets**, there are equivalent (and useful!) definitions of topological spaces by axioms for the **closed sets**, and via axioms for **neighborhoods**. Another important and useful point of view is to view the topological spaces as a category with morphisms the **continuous functions**. From this point of view the notion of *topological space* and the notion of *continuous function* are "equivalent data".

4.6.2 Filters, Hausdorff and Compact spaces

The treatment of filters here is a distillation of material found in Bourbaki: the definition of filter, is in [Bou, Top. Ch. I §6 no. 1], the definition of limit point and cluster point of a filter are [Bou, Top. Ch. I, §7 Def. 1 and 2] and the definition of limit point and cluster point of a function are [Bou, Top. Ch. I §7 Def. 3]. Theorem **??** is [Bou, Top. Ch. I §7 Prop. 9] and Proposition **??** is Example 1 in [Bou, Top. Ch. I §7 no. 3].

The presentation of the equivalent conditions for **Hausdorff spaces**, Theorem 4.7, follows Bourbaki [Bou, Top. Ch. I §8 no. 1].

(H3) The condition that $\Delta(X)$ is closed in $X \times X$ is the condition used in algebraic geometry for a separated scheme (see [Ha, Ch. II §4] and Macdonald (1.11) in [CSM]).

(H5) Hausdorff spaces are the spaces such that limits are unique, when they exist.

(H1) The condition (H1) is the separation axiom that is used often as the definition of a Hausdorff topological space.

The presentation of the equivalent conditions for **compact spaces**, Theorem 4.8, follows [Bou, Top. Ch. I §9 no. 1]. The second and third conditions in the definition of a filter say that finite intersections of elements of a filter cannot be empty. This is the rigidity condition that plays an important role in arguments relating limit points and compactness.

4.7 Some proofs

4.7.1 Cluster and limit points; Cauchy and convergent sequences

The proof of part (e) of Proposition 4.11 uses Proposition 4.12(a).

Proposition 4.11. *Let (X, d) be a metric space. Let $A \subseteq X$ and let $(a_1, a_2, \ldots)$ be a sequence in A.*

(a) *(Limit points are unique) If $z_1, z_2 \in X$ are limit points of $(a_1, a_2, \ldots)$ then $z_1 = z_2$.*

(b) *(Limit points are cluster points) If $z \in X$ is a limit point of $(a_1, a_2, \ldots)$ then z is a cluster point of $(a_1, a_2, \ldots)$.*

(c) *(Cluster points of Cauchy sequences are limit points) If $(a_1, a_2, \ldots)$ is a Cauchy sequence and z is a cluster point of $(a_1, a_2, \ldots)$ then z is a limit point of $(a_1, a_2, \ldots)$.*

(d) *(Convergent sequences are Cauchy) If there exists $z \in X$ such that z is a limit point of $(a_1, a_2, \ldots)$ then $(a_1, a_2, \ldots)$ is Cauchy sequence.*

(e) *If A is ball compact in X then $(a_1, a_2, \ldots)$ has a Cauchy subsequence.*

Proof.

$d(a_{j_m}^{(m)}, a_{j_n}^{(n)}) \le d(a_{j_m}^{(m)}, a_{j_{k+1}}^{(k+1)}) + d(a_{j_{k+1}}^{(k+1)}, a_{j_n}^{(n)}) \le 10^{-(k+1)} + 10^{-(k+1)} \le 10^{k})$.
Let $z \in A$.
To show: z is not a limit point of $(a_{j_1}^{(1)}, a_{j_2}^{(2)}, \ldots)$.
To show: There exists $\epsilon \in \mathbb{E}$ and $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\ge \ell}$ then $d(a_{j_n}^{(n)}, z) > \epsilon$.
Let $U \in \mathcal{S}$ such that $z \in U$.
Since U is open in X then there exists $k \in \mathbb{Z}_{>0}$ such that $B_{10^{-k}}(z) \subseteq U$.
Let $\epsilon = 10^{-k}$ and let $\ell = k$.
To show: If $n \in \mathbb{Z}_{\ge \ell}$ then $d(a_{j_n}^{(n)}, z) > \epsilon$.
Assume $n \in \mathbb{Z}_{\ge \ell}$.
Since $B_{10^{-n}}(a_{j_n}^{(n)}) \not\subseteq B_{10^{-k}}(z)$ there exists $y \in B_{10^{-n}}(a_{j_n}^{(n)})$ such that $d(y, z) > 10^{-k}$.
Thus $d(a_{j_n}^{(n)}, z) \ge d(y, z) - d(a_{j_n}^{(n)}, y) > 10^{-k} - 10^{-n} > 10^{-k} = \epsilon$.
So z is not a limit point of $(a_{j_1}^{(1)}, a_{j_2}^{(2)}, \ldots)$.
So A is not Cauchy compact. □

4.8.9 Subsets of $\mathbb{R}^n$

Proposition 4.22. *Let $\mathbb{R}^n$ have the standard metric and let $A \subseteq \mathbb{R}^n$.*

(a) (Bounded subsets of $\mathbb{R}^n$ are ball compact) If A is bounded then A is ball compact.

(b) (Closed subsets of $\mathbb{R}^n$ are Cauchy compact) If A is closed in $\mathbb{R}^n$ then A is Cauchy compact.

Proof.

(a) Assume $A \subseteq \mathbb{R}^n$ is bounded.
To show: A is ball compact.
To show: If $\epsilon \in \mathbb{E}$ then there exist $x_1, \ldots, x_\ell \in \mathbb{R}^n$ such that $A \subseteq B_\epsilon(x_1) \cup \cdots \cup B_\epsilon(x_\ell)$.
Since A is bounded then there exists $x \in \mathbb{R}_n$ and $M \in \mathbb{R}_{>0}$ such that $A \subseteq B_M(x)$.
Let $J = \{x + (c_1, \ldots, c_n) \in \mathbb{R}^n \mid c_i \in \{k10^{-\ell} \mid k \in \{-M, \ldots, M\}\}$.
Then

$$\Big(\bigcup_{y \in J} B_\epsilon(y)\Big) \supseteq B_M(x) \supseteq A \qquad \text{and} \qquad \mathrm{Card}(J) = (2M)^n.$$

So A is ball compact in $\mathbb{R}^n$. (EXACTLY WHAT PROPERTY OF $\mathbb{R}^n$ DID WE USE?? I THINK THIS IS THE ARCHIMEDEAN PROPERTY)

(b) Assume that A is closed in $\mathbb{R}^n$.
To show: A is Cauchy compact.
Since $\mathbb{R}^n$ is Cauchy compact and A is closed then, by Proposition 4.4(e), A is Cauchy compact. □

4.8.10 Equivalent characterizations of Hausdorff spaces

Theorem 4.23. *Let $(X, \mathcal{T})$ be a topological space. The following are equivalent.*

(H) If $x, y \in X$ and $x \ne y$ then there exist $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$ such that $U \cap V = \emptyset$.

(H1) If $x \in X$ then $\bigcap_{N \in \mathcal{N}(x)} \overline{N} = \{x\}$.

(H2) If $\Delta\colon X \to X \times X$ is the diagonal map then $\Delta(X)$ is closed in $X \times X$.

(H3) If I is a set and $\Delta\colon X \to \prod_{k \in I} X_k$, where $X_k = X$ for $k \in I$, is the diagonal map then $\Delta(X)$ is closed in $\prod_{k \in I} X_k$.

(H4) If $\mathcal{G}$ is a filter on X then $\mathcal{G}$ has at most one limit point.

(H5) If $\mathcal{J}$ is a filter on X and x is a limit point of $\mathcal{J}$ then x is the only cluster point of $\mathcal{J}$.

Proof.
(H3) $\Rightarrow$ (H2): (H2) is a special case of (H3).
(H2) $\Rightarrow$ (H): Assume $x, y \in X$ and $x \neq y$.

> Then $(x, y) \in X \times X$ and $(x, y) \notin \Delta(X)$.
> Thus, by (H2), $(x, y) \notin \overline{\Delta(X)}$.
> So (x, y) is not a close point of $\Delta(X)$.
> So there exists a neighborhood $Z \in \mathcal{N}((x, y))$ such that $Z \cap \Delta(X) = \emptyset$.
> By the definition of the product topology,
> there exist $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$ such that $(U \times V) \cap \Delta(X) = \emptyset$.
> So $U \cap V = \emptyset$.

(H) $\Rightarrow$ (H3):
Assume that if $x, y \in X$ and $x \neq y$ then there exist $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$ such that $U \cap V = \emptyset$.
To show: $\Delta(X)$ is closed in $\prod_{k \in I} X_k$, where $X_k = X$.

> To show: If $x \in \prod_{k \in I} X_k$ and $x \notin \Delta(X)$ then x is not a close point of $\Delta(X)$.
> Assume $x = (x_k) \in \prod_{k \in I} X_k$ and $x \notin \Delta(X)$.
> To show: There exists $W \in \mathcal{N}(x)$ such that $W \cap \Delta(X) = \emptyset$.
> Let $i, j \in I$ such that $x_i \neq x_j$.
> Let $V_i \in \mathcal{N}(x_i)$ and $V_j \in \mathcal{N}(x_j)$ such that $V_i \cap V_j = \emptyset$.
> Then $W = V_i \times V_j \times \prod_{k \neq i,j} X_k \in \mathcal{N}(x)$ and $W \cap \Delta(X) = \emptyset$.
> So x is not a close point on $\Delta(X)$.
> So $\Delta(X)$ is closed in $\prod_{k \in I} X_k$.

(H) $\Rightarrow$ (H1): Assume (H). Assume that if $x, y \in X$ and $x \neq y$ then there exist $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$ such that $U \cap V = \emptyset$.
To show: If $x \in X$ then $\bigcap_{N \in \mathcal{N}(x)} \overline{N} = \{x\}$.

> Assume $x \in X$.
> To show: If $y \in X$ and $y \notin \{x\}$ then $y \notin \bigcap_{N \in \mathcal{N}(x)} \overline{N}$.
> Assume $y \in X$ and $y \notin \{x\}$.
> To show: There exists $U \in \mathcal{N}(x)$ such that $y \notin \overline{U}$.
> By (H), since $y \neq x$, there exist $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$ such that $U \cap V = \emptyset$.
> So there exists $V \in \mathcal{N}(y)$ such that $V \cap U \neq \emptyset$.
> So y is not a close point to U.
> So $y \notin \overline{U}$.
> So $y \notin \bigcap_{N \in \mathcal{N}(x)} \overline{N}$.

(H1) $\Rightarrow$ (H5): Assume that if $x \in X$ then $\bigcap_{N \in \mathcal{N}(x)} \overline{N} = \{x\}$.
To show: If $\mathcal{J}$ is a filter on X and x is a limit point of $\mathcal{J}$ then x is the only cluster point $\mathcal{J}$.

> Assume $\mathcal{J}$ is a filter on X and x is a limit point of $\mathcal{J}$.
> To show: If $y \in X$ is a cluster point of $\mathcal{J}$ then $y = x$.
> Assume $y \in X$ is a cluster point of $\mathcal{J}$.
> Since y is a cluster point of $\mathcal{J}$ then $y \in \bigcap_{M \in \mathcal{J}} \overline{M}$.
> Since x is a limit point of $\mathcal{J}$ then $\mathcal{J} \supseteq \mathcal{N}(x)$.

So
$$y \in \Big(\bigcap_{M\in\mathcal{J}} \overline{M}\Big) \subseteq \Big(\bigcap_{N\in\mathcal{N}(x)} \overline{N}\Big) = \{x\}.$$
So $y = x$.

(H5) $\Rightarrow$ (H4): Assume that if $\mathcal{J}$ is a filter on X and x is a limit point of $\mathcal{J}$ then x is the only cluster point $\mathcal{J}$.
To show: If $\mathcal{G}$ is a filter on X then $\mathcal{G}$ has at most one limit point.

Assume $\mathcal{G}$ is a filter on X and x is a limit point of $\mathcal{G}$.
To show: If $y \in X$ is a limit point of $\mathcal{G}$ then $y = x$.
Assume $y \in X$ is a limit point of $\mathcal{G}$.
Since x is a limit point of $\mathcal{G}$ then $\mathcal{G} \supseteq \mathcal{N}(x)$.
So
$$x \in \Big(\bigcap_{N\in\mathcal{N}(x)} \overline{N}\Big) \supseteq \Big(\bigcap_{M\in\mathcal{G}} \overline{M}\Big).$$
So x is a cluster point of $\mathcal{G}$.
By (H5), y is the only cluster point of $\mathcal{G}$ and so $y = x$.

So $\mathcal{G}$ has at most one limit point.
(H4) $\Rightarrow$ (H): Assume not (H).

Let $x, y \in X$ with $x \neq y$ such that there do not exist $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$ with $U \cap V = \emptyset$.
Let $\mathcal{J}$ be the filter generated by
$$\mathcal{B} = \{U \cap V \mid U \in \mathcal{N}(x), V \in \mathcal{N}(y)\}.$$
Since $X \in \mathcal{N}(y)$ then $\mathcal{N}(x) = \{U \cap X \mid U \in cN(x)\} \subseteq \mathcal{B} \subseteq \mathcal{J}$.
Since $X \in \mathcal{N}(x)$ then $\mathcal{N}(y) = \{X \cap V \mid V \in cN(y)\} \subseteq \mathcal{B} \subseteq \mathcal{J}$.
So x and y are both limit points of $\mathcal{J}$.
Since $x \neq y$ then (H4) does not hold.

□

4.8.11 Equivalent characterizations of compact spaces

Theorem 4.24. *Let $(X, \mathcal{T})$ be a topological space. The following are equivalent.*

(C1) If $\mathcal{J}$ is an filter on X then there exists $x \in X$ such that x is a cluster point of $\mathcal{J}$.

(C2) If $\mathcal{G}$ is an ultrafilter on X then there exists $x \in X$ such that x is a limit point of $\mathcal{G}$.

(C3) If $\mathcal{C}$ is a collection of closed sets such that $\bigcap_{K\in\mathcal{C}} K = \emptyset$ then there exists $\ell \in \mathbb{Z}_{>0}$ and $K_1, K_2, \ldots, K_\ell \in \mathcal{C}$ such that $K_1 \cap K_2 \cap \cdots \cap K_\ell = \emptyset$.

(C4) If $\mathcal{S}$ is a collection of open sets such that $\bigcap_{U\in\mathcal{S}} U = X$ then there exists $\ell \in \mathbb{Z}_{>0}$ and $U_1, U_2, \ldots, U_\ell \in \mathcal{S}$ such that $U_1 \cup U_2 \cup \cdots \cup U_\ell = X$.

Proof. (Sketch)
(C3) $\Leftrightarrow$ (C4) by taking complements.
(C1) $\Rightarrow$ (C2): Assume (C1).