MAST30026 Metric and Hilbert Spaces

Assignment 3

Due: 4pm Thursday October 6, 2022

Question 1. (Product metric gives product topology) Let (X, d_X) and (Y, d_Y) be metric spaces and let $d\colon (X\times Y)\times(X\times Y)\to\mathbb{R}_{\ge 0}$ be defined by

$$d((x_1,y_1),(x_2,y_2)) = d_X(x_1,x_2) + d_Y(y_1,y_2).$$

(a) Prove carefully that d is a metric on $X\times Y$.

(b) Let $\mathbb{E} = \{10^{-1}, 10^{-2}, \ldots\}$ and let $B_\epsilon(p)$ denote the open ball of radius ϵ at p. Using the metrics d_X, d_Y and d define

$$\mathcal{B}_{X\times Y} = \{B_\epsilon(x,y) \mid \epsilon\in\mathbb{E}, (x,y)\in X\times Y\}, \qquad \text{and}$$
$$\mathcal{P}_{X\times Y} = \{B_{\epsilon_1}(x)\times B_{\epsilon_2}(y) \mid \epsilon_1,\epsilon_2\in\mathbb{E}, x\in X, y\in Y\}.$$

Let $\mathcal{T}_m$ be the topology on $X\times Y$ generated by $\mathcal{B}_{X\times Y}$ and let $\mathcal{T}$ be the topology on $X\times Y$ generated by $\mathcal{P}_{X\times Y}$. Prove carefully that $\mathcal{T}_m = \mathcal{T}$.

(c) Carefully sketch the open ball $B_1(0)$ in each of the metric spaces $(\mathbb{R}^3, d_1)$, $(\mathbb{R}^3, d_2)$ and $(\mathbb{R}^3, d_\infty)$, where

$$d_1(x,y) = |x_1-y_1| + |x_2-y_2| + |x_3-y_3|,$$
$$d_2(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2},$$
$$d_\infty = \max\{|x_1-y_1|, |x_2-y_2|, |x_3-y_3|\}.$$

for $x = (x_1,x_2,x_3)$ and $y = (y_1,y_2,y_3)$ in $\mathbb{R}^3$.

Question 2. The *Zariski topology* (also called the *cofinite topology*) on $\mathbb{R}$ is

$$\mathcal{T} = \{U\subseteq\mathbb{R} \mid U^c \text{ is finite}\}\cup\{\emptyset,\mathbb{R}\}, \qquad \text{where } U^c \text{ denotes the complement of } U \text{ in } \mathbb{R}.$$

(a) Prove carefully that $\mathcal{T}$ is a topology on $\mathbb{R}$.

(b) Prove carefully that $(\mathbb{R},\mathcal{T})$ is not Hausdorff.

(c) Determine (with proof) $\overline{\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}}$ in $(\mathbb{R},\mathcal{T})$.

(d) Determine (with proof) the connected sets in $(\mathbb{R},\mathcal{T})$.

(e) Determine (with proof) the compact sets in $(\mathbb{R},\mathcal{T})$.

(f) Find (with proof) a metric $d\colon\mathbb{R}\times\mathbb{R}\to\mathbb{R}_{\ge 0}$ such that the metric space topology on $\mathbb{R}$ determined by the metric d is the same as $\mathcal{T}$.

Question 3. Let $X = \{0_1\}\cup\{0_2\}\cup\mathbb{R}_{>0}$ and define a function $d\colon X\times X\to\mathbb{R}_{\ge 0}\cup\{\infty\}$ by

$$d(y,x) = d(x,y) \qquad \text{and} \qquad d(x,y) = \begin{cases} |y-x|, & \text{if } x,y\in\mathbb{R}_{>0},\\ y, & \text{if } y\in\mathbb{R}_{>0} \text{ and } x\in\{0_1,0_2\},\\ 0, & \text{if } x = 0_1 \text{ and } y = 0_1,\\ 0, & \text{if } x = 0_2 \text{ and } y = 0_2,\\ \infty, & \text{if } x = 0_1 \text{ and } y = 0_2.\end{cases}$$

Let $\mathbb{E} = \{10^{-1}, 10^{-2}, \ldots\}$ and let $B_\epsilon(x) = \{y \in X \mid d(y,x) < \epsilon\}$ for $\epsilon \in \mathbb{E}$ and $x \in X$. Let $\mathcal{T}$ be the topology on X generated by

$$\mathcal{B} = \{B_\epsilon(x) \mid \epsilon \in \mathbb{E}, x \in X\}.$$

(a) Prove carefully that $(X, \mathcal{T})$ is not Hausdorff.

(b) Determine (with proof) $\overline{\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}}$ in $(X, \mathcal{T})$.

Question 4. Let $(X, \mathcal{T})$ be a topological space. Let $S \subseteq X$.

(a) Carefully define what it means for S to be connected.
(b) Carefully define what it means for S to be path connected.
(c) Prove carefully that if S is path connected then S is connected.
(d) Give an explicit example of subset S of $\mathbb{R}^2$ (with the standard metric topology) such that S is connected but S is not path connected. Be sure to prove carefully that the S in your example is connected and is not path connected.

Question 5. Let t be a formal variable and define the *field of formal power series* $\mathbb{C}((t))$ and its *ring of integers* $\mathbb{C}[[t]]$ by

$$\begin{aligned}
&\mathbb{C}((t)) = \{a_k t^k + a_{k+1}t^{k+1} + \cdots \mid k \in \mathbb{Z}, a_i \in \mathbb{C} \text{ and } a_k \neq 0\} \cup \{0\}, \\
&\quad \cup| \\
&\mathbb{C}[[t]] = \{a_0 + a_1 t + a_2 t^2 + \cdots \mid a_i \in \mathbb{C}\}.
\end{aligned}$$

Define functions $|\ |\colon \mathbb{C}((t)) \to \mathbb{R}_{\geq 0}$ and $d\colon \mathbb{C}((t)) \times \mathbb{C}((t)) \to \mathbb{R}_{\geq 0}$ by

$$d(a,b) = |b - a|, \qquad \text{where} \quad |a_k t^k + a_{k+1}t^{k+1} + \cdots| = 10^{-k} \quad \text{if } a_k \neq 0.$$

(a) Prove carefully that

$$\mathbb{C}((t)) = \Big\{ \frac{f(t)}{g(t)} \ \Big|\ f(t), g(t) \in \mathbb{C}[[t]] \text{ and } g(t) \neq 0 \Big\}.$$

(b) Prove carefully that $d\colon \mathbb{C}((t)) \times \mathbb{C}((t)) \to \mathbb{R}_{\geq 0}$ is a metric on $\mathbb{C}((t))$.
(c) Prove that $\mathbb{C}[[t]]$ is the completion of $\mathbb{C}[t]$.
(d) Prove that $\mathbb{C}((t))$ is the completion of $\mathbb{C}(t)$, where

$$\mathbb{C}(t) = \Big\{ \frac{f(t)}{g(t)} \ \Big|\ f(t), g(t) \in \mathbb{C}[t] \text{ and } g(t) \neq 0 \Big\}.$$

(e) For $z(t) \in \mathbb{C}((t))$ define

$$\exp(z(t)) = \sum_{n \in \mathbb{Z}_{>0}} \frac{z(t)^n}{n!}.$$

Prove carefully that the radius of convergence of $\exp(z(t))$ is 1 i.e., prove that if $|z(t)| < 1$ then $\exp(z(t))$ converges to an element of $\mathbb{C}((t))$ and if $|z(t)| > 1$ then $\exp(z(t))$ does not converge to an element of $\mathbb{C}((t))$. Give an example of $z(t) \in \mathbb{C}((t))$ such that $|z(t)| = 1$ and $\exp(z(t))$ does not converge to an element of $\mathbb{C}((t))$, and also give an example of $z(t) \in \mathbb{C}((t))$ with $|z(t)| = 1$ such that $\exp(z(t))$ does converge to an element of $\mathbb{C}((t))$.