16 Lecture 3: Proofs

16.1 Lecture 3: Proof of the glue relations

Proposition 16.1. *(The glue relations) Define*

$$g^\vee = x_1 T_1 \cdots T_{n-1}.$$

Then

$$T_1^{-1} g g^\vee = g^\vee g T_{n-1} \qquad \text{and} \qquad T_{n-1}^{-1} \cdots T_1^{-1} g (g^\vee)^{-1} = q (g^\vee)^{-1} g T_{n-1} \cdots T_1.$$

Proof. Using $x_{i+1} = T_i x_i T_i$ and $x_n g = g x_{n-1}$ and $T_{i+1}^{-1} g = T_i^{-1} g$ from (XaffHeckerelsF) and (DAHArels2F), respectively, gives

$$\begin{aligned}
g^\vee g T_{n-1} &= x_1 T_1 \cdots T_{n-1} g T_{n-1} = T_1^{-1} \cdots T_{n-1}^{-1} T_{n-1} \cdots T_1 x_1 T_1 \cdots T_{n-1} g T_{n-1} \\
&= T_1^{-1} \cdots T_{n-1}^{-1} x_n g T_{n-1} = T_1^{-1} \cdots T_{n-1}^{-1} g x_{n-1} T_{n-1} \\
&= T_1^{-1} g T_1^{-1} \cdots T_{n-2}^{-1} x_{n-1} T_{n-1} = T_1^{-1} g T_1^{-1} \cdots T_{n-2}^{-1} T_{n-2} \cdots T_1 x_1 T_1 \cdots T_{n--2} T_{n-1} \\
&= T_1^{-1} g x_1 T_1 \cdots T_{n-1} = T_1^{-1} g g^\vee.
\end{aligned}$$

Using $q^{-1} g^{-1} x_1 = x_n g^{-1}$ and $x_{i+1} = T_i x_i T_i$ from (periodicityrelsF) and (XaffHeckerelsF), respectively, gives

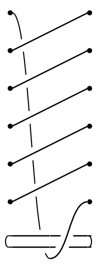
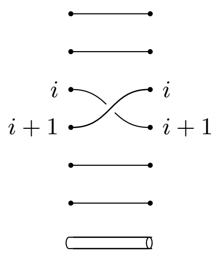
$$\begin{aligned}
q^{-1} T_1^{-1} \cdots T_{n-1}^{-1} g^{-1} g^\vee &= T_1^{-1} \cdots T_{n-1}^{-1} q^{-1} g^{-1} x_1 T_1 \cdots T_{n-1} \\
&= T_1^{-1} \cdots T_{n-1}^{-1} x_n g^{-1} T_1 \cdots T_{n-1} \\
&= T_1^{-1} \cdots T_{n-1}^{-1} T_{n-1} \cdots T_1 x_1 T_1 \cdots T_{n-1} g^{-1} T_1 \cdots T_{n-1} \\
&= x_1 T_1 \cdots T_{n-1} g^{-1} T_1 \cdots T_{n-1} = g^\vee T_1 \cdots T_{n-1},
\end{aligned}$$

which, after taking inverse of both sides is the last relation in the statement. □

16.2 Lecture 3: Proof that the Cherednik-Dunkl operators commute

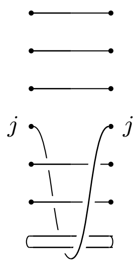
Proposition 16.2. *If $i, j \in \{1, \ldots, n\}$ then $Y_i Y_j = Y_j Y_i$.*

Proof. The group generated by g and $T_0, T_1, \ldots, T_{n-1}$ with the relations (involving T_i and g) from (HeckerelsF) and (DAHArels2F) has a pictorial representation given by

$g =$  and $T_i =$ 

for $i = 1, \ldots, n-1$,

so that

$Y_j = T_{j-1}^{-1} \cdots T_1^{-1} g T_{n-1} \cdots T_j =$ 

for $j \in \{1, \ldots, n\}$. (Yjpicture)

The pictorial representation provides an easy check of the relations

$$Y_iY_j = Y_jY_i, \qquad g^n = Y_1\cdots Y_n \quad \text{and} \quad Y_1Y_n^{-1} = T_0T_{n-1}\cdots T_1\cdots T_{n-1}, \qquad \text{(Ycommutation)}$$

for $i,j\in\{1,\ldots,n\}$. $\square$

16.3 Lecture 3: Proof that the intertwiners intertwine

Proposition 16.3. *If $i\in\{1,\ldots,n-1\}$ and $j\in\{1,\ldots,n\}$ then*

$$\tau_i^\vee Y_i = Y_{i+1}\tau_i^\vee, \qquad \tau_i^\vee Y_{i+1} = Y_i\tau_i^\vee, \qquad \textit{and} \qquad \tau_i^\vee Y_j = Y_j\tau_i^\vee \quad \textit{if } j\notin\{i,i+1\}. \qquad \text{(taupastYrels1B)}$$

If $j\in\{1,\ldots,n-1\}$ then

$$\tau_\pi^\vee Y_j = Y_{j+1}\tau_\pi^\vee \qquad \textit{and} \qquad \tau_\pi^\vee Y_n = q^{-1}Y_1\tau_\pi^\vee. \qquad \text{(taupastYrels2B)}$$

Proof. The relations $Y_{i+1}=T_iY_iT_i$ and $T_i^2=(t^{\frac12}-t^{-\frac12})T_i+1$ and $T_i-T_i^{-1}=t^{\frac12}-t^{-\frac12}$ give

$$\begin{aligned} T_iY_i &= Y_{i+1}T_i^{-1} = Y_{i+1}(T_i-(t^{\frac12}-t^{-\frac12})) = Y_{i+1}T_i-(t^{\frac12}-t^{-\frac12})Y_{i+1},\\ T_iY_{i+1} &= T_i^2Y_iT_i = ((t^{\frac12}-t^{-\frac12})T_i+1)Y_iT_i = Y_iT_i+(t^{\frac12}-t^{-\frac12})Y_{i+1}. \end{aligned}$$

The relations $T_iY_j=Y_jT_i$ for $j\notin\{i,i+1\}$ are most easiy verified by the pictorial technique of the proof of Proposition 3.2. In summary,

$$\begin{aligned} T_iY_i &= Y_{i+1}T_i-(t^{\frac12}-t^{-\frac12})Y_{i+1},\\ T_iY_{i+1} &= Y_iT_i+(t^{\frac12}-t^{-\frac12})Y_{i+1}, \end{aligned} \qquad \text{and} \qquad T_iY_j=Y_jT_i \quad \text{if } j\notin\{i,i+1\}. \qquad \text{(TpastYrelsB)}$$

and from these (and the fact that $Y_1,\ldots,Y_n$ all commute with each other) the statements in (taupastYrels1B) follow.

Using the glue relations (see Proposition 3.1),

$$\begin{aligned} \tau_\pi^\vee Y_1 &= g^\vee gT_{n-1}\cdots T_1 = T_1^{-1}gg^\vee T_{n-2}\cdots T_1 = T_1^{-1}gT_{n-1}\cdots T_2g^\vee\\ &= T_1^{-1}gT_{n-1}\cdots T_2T_1T_1^{-1}g^\vee = T_1^{-1}Y_1T_1^{-1}g^\vee = Y_2g^\vee = Y_2\tau_\pi^\vee, \quad \text{and}\\ \tau_\pi^\vee Y_n &= g^\vee T_{n-1}^{-1}\cdots T_1^{-1}Y_1T_1^{-1}\cdots T_{n-1}^{-1}\\ &= g^\vee T_{n-1}^{-1}\cdots T_1^{-1}gT_{n-1}\cdots T_1T_1^{-1}\cdots T_{n-1}^{-1} = g^\vee T_{n-1}^{-1}\cdots T_1^{-1}g(g^\vee)^{-1}g^\vee\\ &= g^\vee q(g^\vee)^{-1}gT_{n-1}\cdots T_1g^\vee = qY_1g^\vee = qY_1\tau_\pi^\vee, \end{aligned}$$

and if $i>1$ then

$$\tau_\pi^\vee Y_i = g^\vee T_{i-1}^{-1}\cdots T_1^{-1}Y_1T_1\cdots Y_{i-1}^{-1} = T_i^{-1}\cdots T_2^{-1}Y_2T_2^{-1}\cdots T_i^{-1}g^\vee = Y_{i+1}g^\vee = Y_{i+1}\tau_\pi^\vee,$$

which establishes the relations in (taupastYrels2B). $\square$

16.4 Lecture 3: Proof of the polynomial action of the DAHA

Theorem 16.4. *The formulas (divdiffops) define an action of the double affine Hecke algebra on $\mathbb{C}[x_1^{\pm1},\ldots,x_n^{\pm1}]$.*

Proof. The first two identities in (periodicityrelsF) are definitions and the last identity in (periodicityrelsF) is the commutativity of $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

Using the Leibniz rule for ∂_i,

$$\partial_i x_i = 1 + x_{i+1}\partial_i, \qquad \partial_i x_{i+1} = -1 + x_i\partial_i, \qquad \text{and} \qquad \partial_i x_i x_{i+1} = x_i x_{i+1}\partial_i.$$

Thus

$$T_i x_i = t^{-\frac{1}{2}} x_{i+1}\partial_i x_i - t^{\frac{1}{2}}\partial_i x_{i+1} x_i = t^{-\frac{1}{2}} x_{i+1}(1 + x_{i+1}\partial_i) - t^{\frac{1}{2}} x_{i+1} x_i \partial_i$$

is equal to

$$\begin{aligned} x_{i+1}T_i - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})x_{i+1} &= x_{i+1}(t^{-\frac{1}{2}} x_{i+1}\partial_i - t^{\frac{1}{2}}\partial_i x_{i+1}) - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})x_{i+1} \\ &= x_{i+1}(t^{-\frac{1}{2}} x_{i+1}\partial_i - t^{\frac{1}{2}}(-1 + x_i\partial_i) - t^{\frac{1}{2}} + t^{-\frac{1}{2}}). \end{aligned}$$

Similarly show that $T_i x_{i+1} = x_i T_i + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})x_{i+1}$. Since

$$\begin{aligned} T_i(t^{-\frac{1}{2}}\partial_i x_i - t^{\frac{1}{2}} x_i\partial_i) &= (t^{-\frac{1}{2}} x_{i+1}\partial_i - t^{\frac{1}{2}}\partial_i x_{i+1})(t^{-\frac{1}{2}}\partial_i x_i - t^{\frac{1}{2}} x_i\partial_i) = 0 - x_{i+1}\partial_i x_i\partial_i - \partial_i x_{i+1}\partial_i x_i + 0 \\ &= -x_{i+1}(1 + x_{i+1}\partial_i)\partial_i - (-1 + x_i\partial_i)\partial_i x_i = -x_{i+1}\partial_i + \partial_i x_i = 1 \end{aligned}$$

then

$$T_i^{-1} = T_i - (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) = t^{-\frac{1}{2}}(x_{i+1}\partial_i + 1) - t^{\frac{1}{2}}(\partial_i x_{i+1} + 1) = t^{-\frac{1}{2}}\partial_i x_i - t^{\frac{1}{2}} x_i\partial_i.$$

It follows that $T_i^2 = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})T_i + 1$ and $T_i x_i = x_{i+1}T_i^{-1}$ and $x_{i+1} = T_i x_i T_i$.

Let $i \in \{1, \ldots, n-1\}$. Since $y_n x_i = x_i y_n$ and $s_1 \cdots s_{n-1} x_i = x_{i+1} s_1 \cdots s_{n-1}$ then

$$g x_i = s_1 \cdots s_{n-1} y_n x_i = x_{i+1} s_1 \cdots s_{n-1} y_n = x_{i+1} g.$$

As operators on $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$,

$$g x_n = s_1 \cdots s_{n-1} y_n x_n = s_1 \cdots s_{n-1} q^{-1} x_n y_n = q^{-1} x_1 s_1 \cdots s_{n-1} y_n = q^{-1} x_1 g = x_{n+1} g.$$

Let

$$\begin{aligned} s_0 = g s_{n-1} g^{-1} &= (y_1 s_1 \cdots s_{n-1}) s_{n-1} (s_{n-1} \cdots s_1 y_1^{-1}) = y_1 s_1 \cdots s_{n-2} s_{n-1} s_{n-2} \cdots s_1 y_1^{-1} \\ &= y_1 y_n^{-1} s_1 \cdots s_{n-2} s_{n-1} s_{n-2} \cdots s_1. \end{aligned}$$

Then

$$g s_0 g^{-1} = (s_1 \cdots s_{n-1} y_n)(y_1 y_n^{-1} s_{n-1} \cdots s_2 s_1 s_2 \cdots s_{n-1})(s_{n-1} \cdots s_1 y_1^{-1}) = y_2 s_1 y_1^{-1} = s_1.$$

Define

$$\partial_0 = g\partial_{n-1} g^{-1} = g\frac{1}{x_{n-1} - x_n}(1 - s_{n-1})g^{-1} = \frac{1}{x_n - q^{-1}x_1}(1 - s_0).$$

Then

$$g\partial_0 g^{-1} = g\frac{1}{x_n - q^{-1}x_1}(1 - s_0)g^{-1} = \frac{1}{q^{-1}x_1 - q^{-1}x_2}(1 - s_1) = q\partial_1.$$

Hence

$$T_0 = g T_{n-1} g^{-1} = g(t^{-\frac{1}{2}} x_n\partial_{n-1} - t^{\frac{1}{2}}\partial_{n-1} x_n)g^{-1} = t^{-\frac{1}{2}} q^{-1} x_1\partial_0 - t^{\frac{1}{2}}\partial_0 q^{-1} x_n,$$

and

$$g T_0 g^{-1} = g(t^{-\frac{1}{2}} q^{-1} x_1\partial_0 - t^{\frac{1}{2}}\partial_0 q^{-1} x_1)g^{-1} = t^{-\frac{1}{2}} q^{-1} x_2 q\partial_1 - t^{\frac{1}{2}} q\partial_1 q^{-1} x_2 = T_1.$$

□

16.5 Lecture 3: Proof of the various formulas for T_i

The first statement is

$$\begin{aligned}
T_i &= t^{-\frac12}x_{i+1}\partial_i - t^{\frac12}\partial_i x_{i+1} = t^{-\frac12}x_{i+1}\frac{1}{x_i-x_{i+1}}(1-s_i) - t^{\frac12}\frac{1}{x_i-x_{i+1}}(1-s_i)x_{i+1} \\
&= \Big(\frac{(t^{-\frac12}-t^{\frac12})x_{i+1}}{x_i-x_{i+1}}\Big) - \Big(\frac{t^{-\frac12}x_{i+1}-t^{\frac12}x_i}{x_i-x_{i+1}}\Big)s_i = -\frac{(t^{-\frac12}-t^{\frac12})}{1-x_ix_{i+1}^{-1}} + \Big(\frac{t^{-\frac12}-t^{\frac12}x_ix_{i+1}^{-1}}{1-x_ix_{i+1}^{-1}}\Big)s_i \\
&= c_{i,i+1}(x)s_i - (c_{i,i+1}(x)-t^{\frac12}),
\end{aligned}$$

where the last equality follows from

$$c_{i,i+1}(x) - t^{\frac12} = \frac{t^{-\frac12}-t^{\frac12}x_ix_{i+1}^{-1}}{1-x_ix_{i+1}^{-1}} - t^{\frac12} = \frac{t^{-\frac12}-t^{\frac12}x_ix_{i+1}^{-1}-t^{\frac12}+t^{\frac12}x_ix_{i+1}^{-1}}{1-x_ix_{i+1}^{-1}} = \frac{t^{-\frac12}-t^{\frac12}}{1-x_ix_{i+1}^{-1}}$$

Next

$$\begin{aligned}
T_i &= t^{-\frac12}\big(t - \frac{tx_i-x_{i+1}}{x_i-x_{i+1}}(1-s_i)\big) = \frac{t^{\frac12}x_i - t^{\frac12}x_{i+1} - t^{\frac12}x_i + t^{-\frac12}x_{i+1}}{x_i-x_{i+1}} + \frac{t^{\frac12}x_i - t^{-\frac12}x_{i+1}}{x_i-x_{i+1}}s_i \\
&= \frac{t^{-\frac12}x_{i+1}-t^{\frac12}x_{i+1}}{x_i-x_{i+1}} - \frac{t^{-\frac12}x_{i+1}-t^{\frac12}x_i}{x_i-x_{i+1}}s_i = -\frac{t^{-\frac12}-t^{\frac12}}{1-x_ix_{i+1}^{-1}} + \frac{t^{-\frac12}-t^{\frac12}x_ix_{i+1}^{-1}}{1-x_ix_{i+1}^{-1}}s_i \\
&= c_{i,i+1}(x)s_i - (c_{i,i+1}(x)-t^{\frac12}), \qquad \text{proving (TiviaBLop).}
\end{aligned}$$

The formula (TiviacfcnY) is

$$T_i = \tau_i^\vee - \frac{t^{-\frac12}-t^{\frac12}}{1-Y_i^{-1}Y_{i+1}} = \tau_i^\vee - \Big(\frac{t^{-\frac12}-t^{\frac12}Y_i^{-1}Y_{i+1}}{1-Y_i^{-1}Y_{i+1}} - t^{\frac12}\Big) = \tau_i^\vee - (c_{i+1,i}(Y)-t^{\frac12}).$$