7 Lecture 7, 6 April 2022: The Boson Fermion correspondence and the Weyl character formula

7.1 Page 7.1: Geometric Satake

The case $q = 0$ and $t = 0$. The symmetric group acts on $\mathbb{C}[X] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ by permuting the variables. Let $s_1, \ldots, s_{n-1}$ denote the simple reflections in S_n (so that s_i is the transposition switching i and $i+1$) let

$$\mathbb{C}[X]^{S_n} = \{f \in \mathbb{C}[X] \mid \text{if } i \in \{1, \ldots, n-1\} \text{ then } s_i f = f\} \qquad \text{and}$$
$$\mathbb{C}[W]^{\det} = \{f \in \mathbb{C}[X] \mid \text{if } i \in \{1, \ldots, n-1\} \text{ then } s_i f = -f\}.$$

Let

$$p_0 = \sum_{w \in S_n} w \qquad \text{and} \qquad e_0 = \sum_{w \in S_n} (-1)^{\ell(w_0) - \ell(w)} w,$$

where $\ell(w_0) = \frac{1}{2} n(n-1)$. For $\mu \in \mathbb{Z}^n$, the *monomial symmetric function* is

$$m_\mu = \frac{1}{W_\lambda(1)} p_0 x^\mu = \frac{1}{W_\lambda(1)} \sum_{w \in S_n} w x^\mu,$$

where the coefficient $\frac{1}{W_\lambda(1)}$ makes the coefficient of x^μ in m_μ equal to 1. The *skew orbit sum* is

$$a_\mu = e_0 x^\mu = \sum_{w \in S_n} (-1)^{\ell(w_0) - \ell(w)} x^{w\mu} = \det(x_i^{\mu_j}).$$

The special case where $\rho = (n-1, n-2, \ldots, 2, 1, 0)$ gives the *Vandermonde determinant*,

$$a_\rho = (-1)^{\ell(w_0)} \det(x_i^{n-j}) = \prod_{1 \le i < j \le n} (x_j - x_i).$$

If $i \in \{1, \ldots, n-1\}$ then $m_{s_i\mu} = m_\mu$ and $a_\mu = -a_{s_i\mu}$ and so

$$\{m_\lambda \mid \lambda \in (\mathbb{Z}^n)^+\} \quad \text{is a basis of} \quad \mathbb{C}[X]^{S_n} = p_0 \mathbb{C}[X],$$
$$\{a_{\lambda+\rho} \mid \lambda \in (\mathbb{Z}^n)^+\} \quad \text{is a basis of} \quad \mathbb{C}[X]^{\det} = e_0 \mathbb{C}[X],$$

where

$$\begin{matrix} (\mathbb{Z}^n)^+ = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \ge \cdots \ge \lambda_n\} \\ (\mathbb{Z}^n)^{++} = \{\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}^n \mid \gamma_1 > \cdots > \gamma_n\} \end{matrix} \qquad \text{and} \qquad \begin{matrix} (\mathbb{Z}^n)^+ & \xrightarrow{\sim} & (\mathbb{Z}^n)^{++} \\ \lambda & \longmapsto & \lambda + \rho \end{matrix}$$

is a bijection.

For $\lambda \in (\mathbb{Z}^n)^+$, the *Schur function* is

$$s_\lambda = \frac{a_{\lambda+\rho}}{a_\rho}.$$

Schur definitively recognized the function s_λ as the character of a finite dimensional irreducible representation of the group $GL_n(\mathbb{C})$. A way of making the Schur function very natural is to recognize that the following diagram of vector space isomorphisms tells us that $\mathbb{C}[X]^{\det}$ is a free (rank 1) $\mathbb{C}[X]^{S_n}$-module with basis vector a_ρ.

$$\begin{matrix} \mathbb{C}[X]^{W_0} & \xrightarrow{\sim} & \mathbb{C}[X]^{\det} = a_\rho \mathbb{C}[X]^{W_0} \\ f & \longmapsto & a_\rho f \\ s_\lambda & \longmapsto & a_{\lambda+\rho} = e_0 x^{\lambda+\rho} \\ m_\lambda = p_0 x^\lambda & & \end{matrix} \qquad \text{(HWeyl)}$$

Hermann Weyl used this point of view in his generalization of Schur's result which recognized that the analogues of the s_λ for crystallographic reflection groups (Weyl groups) provide the characters of the finite dimensional irreducible representations of compact Lie groups.

The case of $q = 0$ and general t. Let H be the subalgebra of $\widetilde{H}$ generated by $T_1, \ldots, T_{n-1}$ and x_k for $k \in \mathbb{Z}$. The restriction of the polynomial representation $\mathbb{C}[X]$ to the subalgebra H is

$$\mathbb{C}[X] \cong H\mathbf{1}_0 = \mathrm{span}\{x^\mu \mathbf{1}_0 \mid \mu \in \mathbb{Z}^n\}.$$

For $\mu \in \mathfrak{a}^*_{\mathbb{Z}}$ the *Whittaker function*

$$A_\mu(0,t)\mathbf{1}_0 \in \varepsilon_0 H \mathbf{1}_0 \qquad \text{is defined by} \qquad A_\mu(0,t) = \varepsilon_0 X^\mu \mathbf{1}_0.$$

See, for example, [HKP, §6] for the connection between p-adic groups and the affine Hecke algebra and the explanation of why A_μ is equivalent to the data of a (spherical) Whittaker function for a p-adic group. As proved carefully in [NR04, Theorem 2.7], it follows from (2.6) and (2.3) that

$$\varepsilon_0 H \mathbf{1}_0 \quad \text{has } \mathbb{K}\text{-basis} \quad \{A_{\lambda+\rho}(0,t) \mid \lambda \in (\mathbb{Z}^n)^+\}.$$

Following [Lu83] (see [NR04, Theorem 2.4] for another exposition),

$$\begin{array}{lll} \text{the Satake isomorphism,} & \mathbb{K}[X]^{W_0} \cong \mathbf{1}_0 H \mathbf{1}_0, & \text{and} \\ \text{the Casselman-Shalika formula,} & A_{\lambda+\rho}(0,t) = s_\lambda A_\rho, & \end{array}$$

can be formulated by the following diagram of vector space (free K-module) isomorphisms:

$$\begin{array}{ccccc} Z(H) = \mathbb{K}[X]^{W_0} & \xrightarrow{\sim} & \mathbf{1}_0 H \mathbf{1}_0 & \xrightarrow{\sim} & \varepsilon_0 H \mathbf{1}_0 \\ f & \longmapsto & f\mathbf{1}_0 & \longmapsto & A_\rho f \mathbf{1}_0 \\ s_\lambda & \longmapsto & s_\lambda \mathbf{1}_0 & \longmapsto & A_{\lambda+\rho}(0,t) = \varepsilon_0 X^{\lambda+\rho}\mathbf{1}_0 \\ P_\lambda(0,t) & \longmapsto & P_\lambda(0,t)\mathbf{1}_0 = \mathbf{1}_0 X^\lambda \mathbf{1}_0 & & \end{array} \qquad \text{(GeomLang)}$$

As explained by Lusztig [Lu83], in this diagram

$\mathbf{1}_0 H \mathbf{1}_0$ is the *spherical Hecke algebra*

s_λ is the *Schur function*,

$P_\lambda(0,t)$ is the *Hall-Littlewood polynomial*, and

$\{P_\lambda(0,t)\mathbf{1}_0 \mid \lambda \in (\mathbb{Z}^n)^+\}$ is the *Kazhdan-Lusztig basis* of $\mathbf{1}_0 H \mathbf{1}_0$.

The spherical Hecke algebra $\mathbf{1}_0 H \mathbf{1}_0$ is the Iwahori-Hecke algebra corresponding to the *loop Grassmanian* $GL_n(\mathbb{C}((\epsilon)))/GL_n(\mathbb{C}[[t]])$. The statement that $P_\lambda(0,t)\mathbf{1}_0$ is a Kazhdan-Luszitg basis element in $\mathbf{1}_0 H \mathbf{1}_0$ indicates that $P_\lambda(0,t)\mathbf{1}_0$ corresponds to the intersection homology of a Schubert variety in the loop Grassmannian (amazing!).

The diagram (GeomLang) has particular importance due to the fact that $\mathbb{K}[X]^{W_0}$ is an avatar of the Grothendieck group of the category $\mathrm{Rep}(G)$ of finite dimensional representations of G, the spherical Hecke algebra $\mathbf{1}_0 H \mathbf{1}_0$ is a form of the Grothendieck group of K-equivariant perverse sheaves on the loop Grassmanian Gr for the Langlands dual group $G^\vee$, and $\varepsilon_0 H \mathbf{1}_0$ is isomorphic to the Grothendieck group of Whittaker sheaves (appropriately formulated N-equivariant sheaves on Gr); see [FGV].

An analogous picture for general q and general t. The results in Proposition 7.1 and Theorem 7.6 provide an analogous diagram for Macdonald polynomials. Writing the polynomial representation of $\widetilde{H}$ as $\mathbb{C}[X] \cong \widetilde{H}\mathbf{1}_Y$ as in (CXasIndHY), then

$$\begin{array}{ccccc}
\mathbb{C}[X]^{W_0} & \longrightarrow & \mathbb{C}[X]^{W_0}\mathbf{1}_Y = \mathbf{1}_0\widetilde{H}\mathbf{1}_Y & \longrightarrow & A_\rho\mathbb{C}[X]^{W_0} = \varepsilon_0\widetilde{H}\mathbf{1}_Y \\
f & \longmapsto & f\mathbf{1}_Y & \longmapsto & A_\rho f\mathbf{1}_Y \\
P_\lambda(q,qt) & \longmapsto & P_\lambda(q,qt)\mathbf{1}_Y & \longmapsto & A_{\lambda+\rho}(q,t)\mathbf{1}_Y = \varepsilon_0 E_{\lambda+\rho}(q,t)\mathbf{1} \\
P_\lambda(q,t) & \longmapsto & P_\lambda(q,t)\mathbf{1}_Y = \mathbf{1}_0 E_\lambda(q,t)\mathbf{1}_Y & &
\end{array}$$

It would be interesting to understand of this diagram in terms of geometric contexts analogous to those which exists for the $q = 0$ case. Some progress in this direction is found, for example, in Ginzburg-Kapranov-Vasserot [GKV95] and Oblomkov-Yun [OY14].

7.2 Page 7.2: Symmetrizers and the polynomial representation

Let $\mathbb{C}[X] = \mathbb{C}[x_1^{\pm1}, \ldots, x_n^{\pm1}]$. The symmetric group S_n acts on $\mathbb{C}[X]$ by permuting $x_1, \ldots, x_n$. Letting $s_1, \ldots, s_{n-1}$ denote the *simple transpositions* in S_n,

$$(s_i f)(x_1, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_n). \tag{7.1}$$

Define operators $T_1, \ldots, T_{n-1}$ and g on $\mathbb{C}[x_1^{\pm1}, \ldots, x_n^{\pm1}]$ by

$$T_i f = t^{-\frac{1}{2}}\big(t - \frac{tx_i - x_{i+1}}{x_i - x_{i+1}}(1 - s_i)\big)f \tag{7.2}$$

Let $\mathbb{C}[X] = \mathbb{C}[x_1^{\pm1}, \ldots, x_n^{\pm1}]$ and define vector subspaces of $\mathbb{C}[X]$ by

$$\begin{aligned}
\mathbb{C}[X]^{S_n} &= \{f \in \mathbb{C}[X] \mid \text{if } i \in \{1, \ldots, n-1\} \text{ then } s_i f = f\}, \\
\mathbb{C}[X]^{\det} &= \{f \in \mathbb{C}[X] \mid \text{if } i \in \{1, \ldots, n-1\} \text{ then } s_i f = -f\}, \\
\mathbb{C}[X]^{\mathrm{Bos}} &= \{f \in \mathbb{C}[X] \mid \text{if } i \in \{1, \ldots, n-1\} \text{ then } T_{s_i} f = t^{\frac{1}{2}} f\}, \\
\mathbb{C}[X]^{\mathrm{Fer}} &= \{f \in \mathbb{C}[X] \mid \text{if } i \in \{1, \ldots, n-1\} \text{ then } T_{s_i} f = -t^{-\frac{1}{2}} f\},
\end{aligned}$$

Proposition 7.1 shows that there are $\mathbb{C}[X]^{S_n}$-module isomorphisms

$$\begin{array}{ccc}
\mathbb{C}[X]^{S_n} & \to & \mathbb{C}[X]^{\det} \\
f & \mapsto & a_\rho f
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
\mathbb{C}[X]^{\mathrm{Bos}} & \to & \mathbb{C}[X]^{\mathrm{Fer}} \\
f & \mapsto & A_\rho f
\end{array}
\tag{BosFermmaps}$$

where

$$a_\rho = \prod_{1\le i<j\le n} (x_j - x_i) \qquad \text{and} \qquad A_\rho = \prod_{1\le i<j\le n} (x_j - tx_i), \tag{arhoArhodefn}$$

(so that $a_\rho \in \mathbb{C}[X]^{\det}$, $A_\rho \in \mathbb{C}[X]^{\mathrm{Fer}}$ and the coefficient of $x_1^0 x_2 x_3^2 \cdots x_n^{n-1}$ is 1 in both a_ρ and A_ρ). The maps in (BosFermmaps) are *Boson-Fermion correspondences.*

Let

$$p_0 = \sum_{w\in S_n} w \qquad \text{and} \qquad e_0 = \sum_{w\in S_n} (-1)^{\ell(w)} w. \tag{symms}$$

Let $z \in S_n$. A *reduced expression for* z is an expression for z as a product of s_i,

$$z = s_{i_1} \cdots s_{i_\ell}, \qquad \text{such that } i_1, \ldots, i_\ell \in \{1, \ldots, n-1\} \text{ and } \ell = \ell(z).$$

Define

$$T_z = T_{i_1} \cdots T_{i_\ell} \qquad \text{if } z = s_{i_1} \cdots s_{i_\ell} \text{ is a reduced word for } z.$$

The *bosonic symmetrizer* and the *fermionic symmetrizer* are

$$\mathbf{1}_0 = \sum_{z\in S_n} t^{\frac{1}{2}(\ell(z)-\ell(w_0))}T_z \qquad \text{and} \qquad \varepsilon_0 = \sum_{w\in S_n} (-t^{-\frac{1}{2}})^{\ell(z)-\ell(w_0)}T_z. \qquad \text{(bosfersymm)}$$

The bosonic symmetrizer $\mathbf{1}_0$ and the fermionic symmetrizer ε_0 are t-analogues of p_0 and e_0, respectively.

Proposition 7.1. *With notations as in (BosFermmaps), (symms) and (bosfersymm),*

$$p_0\mathbb{C}[X] = \mathbb{C}[X]^{S_n}, \qquad e_0\mathbb{C}[X] = \mathbb{C}[X]^{\det} = a_\rho\mathbb{C}[X]^{S_n} \quad \text{and} \quad a_\rho = e_0x^\rho,$$
$$\mathbf{1}_0\mathbb{C}[X] = \mathbb{C}[X]^{\mathrm{Bos}} = \mathbb{C}[X]^{S_n}, \qquad \varepsilon_0\mathbb{C}[X] = \mathbb{C}[X]^{\mathrm{Fer}} = A_\rho\mathbb{C}[X]^{S_n} \quad \text{and} \quad A_\rho = \varepsilon_0x^\rho,$$

7.3 Page 7.3: The inner product $(,)_{q,t}$

Let $\mathbb{C}[X] = \mathbb{C}[x_1^{\pm1},\ldots,x_n^{\pm1}]$. Define an involution $\overline{}\colon \mathbb{C}[X]\to\mathbb{C}[X]$ by

$$\overline{f}(x_1,\ldots,x_n;q,t) = f(x_1^{-1},\ldots,x_n^{-1};q^{-1},t^{-1}), \qquad \text{(keyinvdefn)}$$

Define

$$\nabla_{q,t} = \prod_{i\neq j}\frac{(x_ix_j^{-1};q)_\infty}{(tx_ix_j^{-1};q)_\infty} \qquad \text{and} \qquad \Delta_{q,t} = \nabla_{q,t}\prod_{i<j}\frac{1-tx_ix_j^{-1}}{1-x_ix_j^{-1}}. \qquad \text{(DnabladefnGL)}$$

Define a scalar product $(\ ,\ )_{q,t}\colon \mathbb{C}[X]\times\mathbb{C}[X]\to\mathbb{C}(q,t)$ by

$$(f_1,f_2)_{q,t} = \mathrm{ct}(f_1\overline{f_2}\Delta_{q,t}), \qquad \text{where} \quad \mathrm{ct}(f) = (\text{constant term in } f), \qquad \text{(innproddefnA)}$$

for $f\in\mathbb{C}[x_1^{\pm1},\ldots,x_n^{\pm1}]$.

Proposition 7.2 shows that, in a suitable sense, the inner product $(,)_{q,t}$ is nondegenerate and normalized Hermitian.

Proposition 7.2.

(a) *(sesquilinear) If $f,g\in\mathbb{C}[X]$ and $c\in\mathbb{C}[q^{\pm1}]$ then*

$$(cf,g)_{q,t} = c(f,g)_{q,t}, \qquad \text{and} \qquad (f,cg)_{q,t} = \overline{c}(f,g)_{q,t}.$$

(b) *(nonisotropy) If $f\in\mathbb{C}[X]$ and $f\neq0$ then $(f,f)_{q,t}\neq0$.*

(c) *(nondegeneracy) If F is a subspace of $\mathbb{C}[X]$ and $(,)_F\colon F\times F\to\mathbb{C}$ is the restriction of $(,)_{q,t}$ to F, then $(,)_F$ is nondegenerate.*

(d) *(normalized Hermitian) If $f_1,f_2\in\mathbb{C}[X]$ then*

$$\frac{(f_2,f_1)_{q,t}}{(1,1)_{q,t}} = \overline{\Big(\frac{(f_1,f_2)_{q,t}}{(1,1)_{q,t}}\Big)}.$$

7.4 Page 7.4: The inner product characterization of E_μ and P_λ

Let $\mu\in\mathbb{Z}^n$. Write

$$x^\mu = x_1^{\mu_1}\cdots x_n^{\mu_n} \qquad \text{if } \mu=(\mu_1,\ldots,\mu_n).$$

Proposition 7.3. *Let $\mu\in\mathbb{Z}^n$. The nonsymmetric Macdonald polynomial E_μ is the unique element of $\mathbb{C}[x_1^{\pm1},\ldots,x_n^{\pm1}]$ such that*

(a) *$E_\mu = x^\mu + $ (lower terms);*

(b) If $\nu \in \mathbb{Z}^n$ and $\nu < \mu$ then $(E_\mu, x^\nu)_{q,t} = 0$.

Define
$$(\mathbb{Z}^n)^+ = \{(\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}^n \mid \gamma_1 \geq \cdots \geq \gamma_n\}.$$
For $\gamma \in (\mathbb{Z}^n)^+$, define the *monomial symmetric function* m_γ by

$$m_\gamma = \sum_{\mu \in S_n \gamma} x^\mu, \qquad \text{where the sum is over all distinct rearrangements of } \gamma.$$

Proposition 7.4. *Let $\lambda \in (\mathbb{Z}^n)^+$. The symmetric Macdonald polynomial P_λ is the unique element of $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{S_n}$ such that*

(a) $P_\lambda = m_\lambda + (\text{lower terms})$;

(b) If $\gamma \in (\mathbb{Z}^n)^+$ and $\gamma < \lambda$ then $(P_\lambda, m_\gamma)_{q,t} = 0$.

7.5 Page 7.5: Going up a level from t to qt

As in (arhoArhodefn) and (slicksymmA), let

$$A_\rho = \prod_{1 \leq i < j \leq n} (x_j - t x_i) \qquad \text{and} \qquad W_0(t) = \sum_{w \in S_n} t^{\ell(w)}.$$

Proposition 7.5. *Let $f, g \in \mathbb{C}[X]^{S_n}$ so that f and g are symmetric functions. Then*

$$(f, g)_{q,qt} = \frac{W_0(qt)}{W_0(t^{-1})} (A_\rho f, A_\rho g)_{q,t}.$$

7.6 Page 7.6: Weyl character formula for Macdonald polynomials

Theorem 7.6. *Let $\lambda \in \mathbb{Z}^n$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then*

$$P_\lambda(q, qt) = \frac{A_{\lambda+\rho}(q,t)}{A_\rho(t)}.$$