5 Lecture 5, 23 March 2022: Principal specializations and hook formulas

5.1 Page 1: Principal specialization formulas

An n-periodic permutation is a bijection $w\colon \mathbb{Z}\to\mathbb{Z}$ such that if $i\in\mathbb{Z}$ then $w(i+n)=w(i)+n$. For an n-periodic permutation w, define

$$\mathrm{Inv}(w)=\left\{(i,k)\ \Big|\ \begin{matrix} i\in\{1,\ldots,n\}, k\in\mathbb{Z}\\ i<k \text{ and } w(i)>w(k)\end{matrix}\right\}\qquad\text{and}\qquad \ell(w)=\#\mathrm{Inv}(w).$$

Define an action of n-periodic permutations on $\mathbb{Z}^n$ by setting

$$w(\mu_1,\ldots,\mu_n)=(\mu_{v(1)}+\ell_1,\ldots,\mu_{v(n)}+\ell_n),$$

where $v(i)\in\{1,\ldots,n\}$ and $\ell_i\in\mathbb{Z}$ are determined by $w(i)=v(i)+\ell_i n$. For $\mu\in\mathbb{Z}^n$ let

u_μ be the minimal length n-periodic permutation such that $u_\mu(0,\ldots,0)=(\mu_1,\ldots,\mu_n)$,
t_μ be the n-periodic permutation given by $t_\mu(i)=i+n\mu_i$, and
$v_\mu\in S_n$ be the minimal length permutation such that $v_\mu\mu$ is weakly increasing.

An explicit formula for the permutation v_μ is given by

$$v_\mu(r)=1+\#\{r'\in\{1,\ldots,r-1\}\mid \mu_{r'}\le\mu_r\}+\#\{r'\in\{r+1,\ldots,n\}\mid \mu_{r'}<\mu_r\}.$$

for $r\in\{1,\ldots,n\}$.

For $i,j\in\{1,\ldots,n\}$ and $\ell\in\mathbb{Z}$ define

$$c_{(i,j+\ell n)}(Y^{-1})=t^{-\frac12}\frac{1-q^\ell tY_i^{-1}Y_j}{1-q^\ell Y_i^{-1}Y_j}\qquad\text{and}\qquad c_w(Y^{-1})=\prod_{(i,k)\in\mathrm{Inv}(w)}c_{(i,k)}(Y^{-1}).$$

Define ring homomorphisms $\mathrm{ev}_0^t\colon\mathbb{C}[Y]\to\mathbb{C}$ and $\mathrm{ev}_0^{t^{-1}}\colon\mathbb{C}[Y]\to\mathbb{C}$ by

$$\mathrm{ev}_0^t(Y_i)=t^{-(i-1)+\frac12(n-1)}\qquad\text{and}\qquad \mathrm{ev}_0^{t^{-1}}(Y_i)=t^{(i-1)-\frac12(n-1)},\qquad\text{for } i\in\{1,\ldots,n\}.$$

Theorem 5.1. *Let $\mu,\lambda\in\mathbb{Z}^n$ with $\lambda_1\ge\cdots\ge\lambda_n$ and let $E_\mu(x_1,\ldots,x_n;q,t)$ and $P_\lambda(x_1,\ldots,x_n;q,t)$ and $A_{\lambda+\rho}(x_1,\ldots,x_n;q,t)$ be the corresponding nonsymmetric, symmetric and fermionic Macdonald polynomials, respectively. Then*

$$\begin{aligned} E_\mu(1,t,t^2,\ldots,t^{n-1};q,t)&=t^{\frac{(n-1)}{2}|\lambda|}t^{-\frac12\ell(v_\mu^{-1})}\mathrm{ev}_0^t(c_{u_\mu}(Y^{-1})),\\ P_\lambda(1,t,t^2,\ldots,t^{n-1};q,t)&=t^{\frac{(n-1)}{2}|\lambda|}\mathrm{ev}_0^{t^{-1}}(c_{t_\lambda}(Y^{-1}))\qquad\textit{and}\\ A_{\lambda+\rho}(1,t,t^2,\ldots,t^{n-1};q,t)&=0,\end{aligned}$$

where $|\mu|=\mu_1+\cdots+\mu_n$.

Let $\mu\in\mathbb{Z}^n_{\ge0}$. For $r\in\{1,\ldots,n\}$ and $c\in\{1,\ldots,\mu_r\}$ define

$$u_\mu(r,c)=\#\{r'\in\{1,\ldots,r-1\}\mid\mu_{r'}<c\le\mu_r\}+\#\{r'\in\{r+1,\ldots,n\}\mid\mu_{r'}<c-1<\mu_r\}).$$

Using the formulas

$$\mathrm{Inv}(t_\lambda)=\bigcup_{i<j}\ \bigcup_{\ell=0}^{\lambda_j-\lambda_i-1}\{\varepsilon_i^\vee-\varepsilon_j^\vee+\ell K\}\qquad\text{and}\qquad\text{(Invfortlambda)}$$

$$\mathrm{Inv}(u_\mu) = \bigcup_{(r,c)\in\mu} \bigcup_{i=1}^{u_\mu(r,c)} \{\varepsilon^\vee_{v_\mu(r)} - \varepsilon^\vee_i + (\mu_r - c + 1)K\} \qquad \text{(Invforumu)}$$

gives the following corollary.

Corollary 5.2. *Let $\mu \in \mathbb{Z}^n_{\geq 0}$ and let λ be the decreasing rearrangement of μ. Let $n(\lambda) = \sum_{i=1}^n (i-1)\lambda_i$. Then*

$$P_\lambda(1, t, t^2, \ldots, t^{n-1}; q, t) = t^{n(\lambda)} \prod_{1\leq i<j\leq n} \prod_{\ell=0}^{\lambda_i-\lambda_j-1} \frac{1 - q^\ell t^{j-i+1}}{1 - q^\ell t^{j-i}}$$

and

$$E_\mu(1, t, t^2, \ldots, t^{n-1}; q, t) = t^{-\frac{1}{2}\ell(v_\mu^{-1})} \prod_{(r,c)\in\mu} \prod_{i=1}^{u_\mu(r,c)} \frac{1 - q^{\mu_r-c+1} t^{v_\mu(r)-i+1}}{1 - q^{\mu_r-c+1} t^{v_\mu(r)-i}}.$$

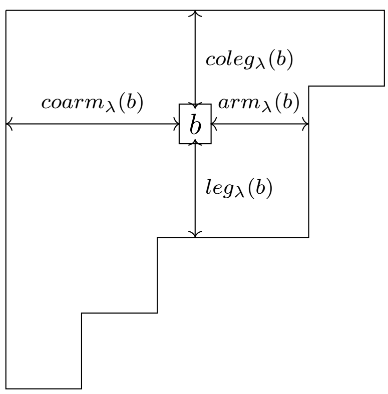
5.2 Page 2: A hook formula for the symmetric case

Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n_{\geq 0}$ with $\lambda_1 \geq \cdots \geq \lambda_n$ and define

$$|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n \qquad \text{and} \qquad n(\lambda) = \sum_{i=1}^n (i-1)\lambda_i.$$

Let λ' denote the conjugate partition to λ (i.e. for $c \in \mathbb{Z}_{>0}$ let $\lambda'_c = \#\{j \in \mathbb{Z}_{>0} \mid \lambda_j \geq c\}$). A *box in* λ is a pair $b = (r, c)$ with $r \in \{1, \ldots, n\}$ and $c \in \{1, \ldots, \lambda_i\}$.

For a box $b = (r, c)$ in λ define



$$\mathrm{coleg}_\lambda(b) = r - 1,$$

$$\mathrm{coarm}_\lambda(b) = c - 1, \qquad b = (r, c), \qquad \mathrm{arm}_\lambda(b) = \lambda_r - c,$$

$$\mathrm{leg}_\lambda(b) = \lambda'_c - r.$$

The *hook length* $h(b)$ and the *content* $c(b)$ of the box b are defined by

$$h(b) = \mathrm{arm}_\lambda(b) + \mathrm{leg}_\lambda(b) + 1 \qquad \text{and} \qquad c(b) = \mathrm{coarm}_\lambda(b) - \mathrm{coleg}_\lambda(b). \qquad \text{(hbcb)}$$

Theorem 5.3. *Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n_{\geq 0}$ with $\lambda_1 \geq \cdots \geq \lambda_n$. Then*

$$P_\lambda(1, t, t^2, \ldots, t^{n-1}; q, t) = t^{n(\lambda)} \prod_{b\in\lambda} \frac{1 - q^{\mathrm{coarm}_\lambda(b)} t^{n-\mathrm{coleg}_\lambda(b)}}{1 - q^{\mathrm{arm}_\lambda(b)} t^{\mathrm{leg}_\lambda(b)+1}}.$$

5.3 Page 3: A hook formula for the nonsymmetric case

Theorem 5.4. *Let* $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n_{\geq 0}$ *and let* λ *be the weakly decreasing rearrangment of* μ*. For* $r \in \{1, \ldots, n\}$ *and* $c \in \{1, \ldots, \mu_r\}$ *define*

$$u_\mu(r,c) = \#\{r' \in \{1,\ldots,r-1\} \mid \mu_{r'} < c \leq \mu_r\} + \#\{r' \in \{r+1,\ldots,n\} \mid \mu_{r'} < c-1 < \mu_r\}) \quad \textit{and}$$
$$v_\mu(r) = 1 + \#\{r' \in \{1,\ldots,r-1\} \mid \mu_{r'} \leq \mu_r\} + \#\{r' \in \{r+1,\ldots,n\} \mid \mu_{r'} < \mu_r\}.$$

Let $n(\lambda) = \sum_{i=1}^n (i-1)\lambda_i$*. Then*

$$E_\mu(1,t,t^2,\ldots,t^{n-1};q,t) = t^{n(\lambda)} \prod_{(r,c)\in\mu} \frac{1-q^c t^{v_\mu(r)}}{1-q^{\mu_r-c+1} t^{v_\mu(r)-u_\mu(r,c)}}.$$

5.4 Page 4: Elliptic, quantum and ordinary dimension formulas

The Schur function s_λ is the specialization of P_λ at $q=t$,

$$s_\lambda(x_1,\ldots,x_n) = P_\lambda(x_1,\ldots,x_n;t,t).$$

Specializing Theorem 5.3 at $q=t$ gives

$$s_\lambda(1,t,t^2,\ldots,t^{n-1}) = \prod_{b\in\lambda} \frac{1-t^{n+c(b)}}{1-t^{h(b)}}, \qquad \text{(qdimLlambda)}$$

where, as defined in (hbcb), $h(b)$ is the hook length of the box b and $c(b)$ is the content of the box b. Setting $t=1$ in (qdimLlambda) gives

$$s_\lambda(1,1,\ldots,1) = \prod_{b\in\lambda} \frac{n+c(b)}{h(b)}. \qquad \text{(dimLlambda)}$$

Specializing the first identity of Corollary 5.2 at $q=t$ gives

$$s_\lambda(1,t,t^2,\ldots,t^{n-1}) = \prod_{1\leq i<j\leq n} \frac{1-t^{\lambda_i-\lambda_j+(j-i)}}{1-t^{j-i}}. \qquad \text{(WqdimLlambda)}$$

Setting $t=1$ in (WqdimLlambda) gives

$$s_\lambda(1,1,\ldots,1) = \prod_{1\leq i<j\leq n} \frac{\lambda_i-\lambda_j+j-i}{j-i}. \qquad \text{(WdimLlambda)}$$

These are special cases of Weyl's integral formula and Weyl's dimension formula (see Bröcker-TomDieck ???).

Let $\mathrm{char}(L(\lambda))$ denote the character of the irreducible polynomial representation of $GL_n(\mathbb{C})$ indexed by λ (see [Mac, Ch. I App. A (8.4)]). By the Weyl character formula (see [Kac, Theorem 10.4])

$$s_\lambda(x_1,\ldots,x_n) = \mathrm{char}(L(\lambda)) = \mathrm{Tr}(L(\lambda), e^x).$$

The specialization

$$s_\lambda(1,t,t^2,\ldots,t^n) = \mathrm{Tr}(L(\lambda), e^\rho) = \mathrm{qdim}(L(\lambda))$$

is the *quantum dimension* of $L(\lambda)$ (see [Kac, Prop, 10.10]). The specialization

$$s_\lambda(1,1,\ldots,1) = \mathrm{Tr}(L(\lambda),1) = \dim(L(\lambda))$$

is the dimension of $L(\lambda)$ (see [Kac, Cor. 10.10]). It would be interesting to give an interpretation of the formula from Corollary 5.2,

$$P_\lambda(1, t, t^2, \ldots, t^n; q, t) = \prod_{1\le i<j\le n} \prod_{r=0}^{\lambda_i-\lambda_j-1} \frac{1-q^r t^{j-i+1}}{1-t^{j-i}},$$

as an "elliptic' dimension" formula for $L(\lambda)$.