4 Lecture 4, 16 March 2022: Symmetrizers and E-expansions

4.1 Page 1: Nonsymmetric, relative, symmetric and fermionic Macdonald polynomials

Let $q, t^{\frac{1}{2}} \in \mathbb{C}^\times$. Let y_n be the operator on $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ given by

$$(y_n h)(x_1, \ldots, x_n) = h(x_1, \ldots, x_{n-1}, q^{-1} x_n).$$

The symmetric group S_n acts on $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ by permuting the variables $x_1, \ldots, x_n$. Define operators $T_1, \ldots, T_{n-1}$, g and $g^\vee$ on $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ by

$$T_i = t^{-\frac{1}{2}}\big(t - \frac{tx_i - x_{i+1}}{x_i - x_{i+1}}(1 - s_i)\big), \qquad g = s_1 s_2 \cdots s_{n-1} y_n, \qquad g^\vee = x_1 T_1 \cdots T_{n-1}, \tag{4.1}$$

where $s_1, \ldots, s_{n-1}$ are the simple transpositions in S_n. The *Cherednik-Dunkl operators* are

$$Y_1 = gT_{n-1} \cdots T_1, \quad Y_2 = T_1^{-1} Y_1 T_1^{-1}, \quad Y_3 = T_2^{-1} Y_2 T_2^{-1}, \quad \ldots, \quad Y_n = T_{n-1}^{-1} Y_{n-1} T_n^{-1}. \tag{4.2}$$

For $\mu \in \mathbb{Z}^n$ the *nonsymmetric Macdonald polynomial* E_μ is the (unique) element $E_\mu \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ such that

$$Y_i E_\mu = q^{-\mu_i} t^{-(v_\mu(i)-1)+\frac{1}{2}(n-1)} E_\mu, \qquad \text{and the coefficient of } x_1^{\mu_1} \cdots x_n^{\mu_n} \text{ in } E_\mu \text{ is } 1, \tag{4.3}$$

where $v_\mu \in S_n$ is the minimal length permutation such that $v_\mu \mu$ is weakly increasing.
Let $\mu = (\mu_1, \ldots, \mu_n)$ and let $z \in S_n$.

$$\text{The } \textit{relative Macdonald polynomial } E_\mu^z \text{ is} \qquad E_\mu^z = t^{-\frac{1}{2}(\ell(zv_\mu^{-1}) - \ell(v_\mu^{-1}))} T_z E_\mu. \tag{4.4}$$

Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n) \in \mathbb{Z}^n$.

$$\text{The } \textit{symmetric Macdonald polynomial } P_\lambda \text{ is} \qquad P_\lambda = \sum_{\nu \in S_n \lambda} t^{\frac{1}{2}\ell(z_\nu)} T_{z_\nu} E_\lambda, \tag{4.5}$$

where the sum is over rearrangements ν of λ and $z_\nu \in S_n$ is minimal length such that $\nu = z_\nu \lambda$.
Let $\rho = (n-1, n-2, \ldots, 2, 1, 0)$. The *fermionic Macdonald polynomial* $A_{\lambda+\rho}$ is

$$A_{\lambda+\rho} = (-t)^{\ell(w_0)} \sum_{z \in S_n(\lambda+\rho)} (-t^{-\frac{1}{2}})^{\ell(z)} T_z E_{\lambda+\rho}. \tag{4.6}$$

4.2 Page 2: H_Y-decomposition of the polynomial representation

Let H_Y be the algebra generated by the operators $T_1, \ldots, T_{n-1}$ and $Y_1, \ldots, Y_n$ (so that H_Y is an affine Hecke algebra). For $i \in \{1, \ldots, n-1\}$, let

$$\tau_i^\vee = T_i + \frac{t^{-\frac{1}{2}}(1-t)}{1 - Y_i^{-1} Y_{i+1}} = T_i^{-1} + \frac{t^{-\frac{1}{2}}(1-t) Y_i^{-1} Y_{i+1}}{1 - Y_i^{-1} Y_{i+1}}, \tag{tauipm}$$

where the second equality is a consequence of $T_i - T_i^{-1} = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$. As H_Y-modules

$$\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] = \bigoplus_\lambda \mathbb{C}[X]^\lambda \qquad \text{where} \qquad \mathbb{C}[X]^\lambda = \text{span}\{E_\mu \mid \mu \in S_n \lambda\},$$

and the direct sum is over decreasing $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n) \in \mathbb{Z}^n$. A description of the action of H on $\mathbb{C}[X]^\lambda$ is given by the following. Let $\mu \in \mathbb{Z}^n$ and let $i \in \{1, \ldots, n-1\}$. Let $v_\mu \in S_n$ be the minimal length permutation such that $v_\mu \mu$ is weakly increasing and let

$$\begin{matrix} a_\mu = q^{\mu_i - \mu_{i+1}} t^{v_\mu(i) - v_\mu(i+1)}, \\ a_{s_i\mu} = q^{\mu_{i+1} - \mu_i} t^{v_\mu(i+1) - v_\mu(i)}, \end{matrix} \qquad \text{and} \qquad D_\mu = \frac{(1 - t a_\mu)(1 - t a_{s_i\mu})}{(1 - a_\mu)(1 - a_{s_i\mu})}. \qquad \text{(forHaction)}$$

Assume that $\mu_i > \mu_{i+1}$. By using the identity $E_{s_i\mu} = t^{\frac{1}{2}} \tau_i^\vee E_\mu$ if $\mu_i > \mu_{i+1}$ from (E2), the eigenvalue from (8.3) and the two formulas in (tauipm),

$$\begin{matrix} Y_i^{-1} Y_{i+1} E_\mu = a_\mu E_\mu, \\ Y_i^{-1} Y_{i+1} E_{s_i\mu} = a_{s_i\mu} E_{s_i\mu}, \end{matrix} \qquad \begin{matrix} t^{\frac{1}{2}} \tau_i^\vee E_\mu = E_{s_i\mu}, \\ t^{\frac{1}{2}} \tau_i^\vee E_{s_i\mu} = D_\mu E_\mu, \end{matrix} \qquad \text{and} \qquad \begin{matrix} t^{\frac{1}{2}} T_i E_\mu = -\frac{1-t}{1-a_\mu} E_\mu + E_{s_i\mu}, \\ t^{\frac{1}{2}} T_i E_{s_i\mu} = D_\mu E_\mu + \frac{1-t}{1-a_{s_i\mu}} E_{s_i\mu}. \end{matrix}$$

(CXlambdaaction)

Now assume that $\mu_i = \mu_{i+1}$. Then $v_\mu(i+1) = v_\mu(i) + 1$ and $a_\mu = t^{-1}$ so that

$$Y_i^{-1} Y_{i+1} E_\mu = t^{-1} E_\mu, \qquad (t^{\frac{1}{2}} \tau_i^\vee) E_\mu = 0, \qquad \text{and} \qquad (t^{\frac{1}{2}} T_i) E_\mu = t E_\mu. \qquad \text{(Tigivest)}$$

These formulas make explicit the action of H_Y on $\mathbb{C}[X]^\lambda$ in the basis $\{E_\mu \mid \mu \in S_n \lambda\}$.

4.3 Page 3: Symmetrizers

4.3.1 Bosonic and fermionic symmetrizers

Let w_0 be the longest element of S_n so that

$$w_0(i) = n - i + 1, \text{ for } i \in \{1, \ldots, n\}, \qquad \text{and} \qquad \ell(w_0) = \frac{n(n-1)}{2} = \binom{n}{2}.$$

Let $z \in S_n$. A *reduced expression for* z is an expression for z as a product of s_i,

$$z = s_{i_1} \cdots s_{i_\ell}, \qquad \text{such that } i_1, \ldots, i_\ell \in \{1, \ldots, n-1\} \text{ and } \ell = \ell(z).$$

Define

$$T_z = T_{i_1} \cdots T_{i_\ell} \qquad \text{if } z = s_{i_1} \cdots s_{i_\ell} \text{ is a reduced word for } z.$$

The *bosonic symmetrizer*

$$\mathbf{1}_0 = \sum_{z \in S_n} t^{\frac{1}{2}(\ell(z) - \ell(w_0))} T_z \qquad \text{is a } t\text{-analogue of} \qquad p_0 = \sum_{z \in S_n} z. \qquad \text{(fullsymm)}$$

The *fermionic symmetrizer*

$$\varepsilon_0 = \sum_{w \in S_n} (-t^{-\frac{1}{2}})^{\ell(z) - \ell(w_0)} T_z \qquad \text{is a } t\text{-analogue of} \qquad e_0 = \sum_{w \in S_n} (-1)^{\ell(z) - \ell(w_0)} z.$$

The symmetrizers satisfy

$$T_i \mathbf{1}_0 = \mathbf{1}_0 T_i = t^{\frac{1}{2}} \mathbf{1}_0 \qquad \text{and} \qquad T_i \varepsilon_0 = \varepsilon_0 T_i = -t^{-\frac{1}{2}} \varepsilon_0, \quad \text{for } i \in \{1, \ldots, n-1\},$$

$$\mathbf{1}_0^2 = t^{-\frac{1}{2}\ell(w_0)} W_0(t) \mathbf{1}_0 \qquad \text{and} \qquad \varepsilon_0^2 = t^{-\frac{1}{2}\ell(w_0)} W_0(t) \varepsilon_0, \qquad \text{(symmprops)}$$

where

$$W_0(t) = \sum_{z \in S_n} t^{\ell(z)} \text{is the } \textit{Poincaré polynomial for } S_n. \qquad \text{(Poincarepoly)}$$

Proposition 4.1. *As operators on* $\mathbb{C}[x_1^{\pm1}, \ldots, x_n^{-1}]$,

$$\mathbf{1}_0 = \Big(\sum_{z\in W} z\Big)\Big(\prod_{1\le i<j\le n} \frac{x_i - tx_j}{x_i - x_j}\Big).$$

Let

$$c_{ij}(x) = \frac{t^{-\frac12} - t^{\frac12}x_ix_j^{-1}}{1 - x_ix_j^{-1}} = t^{-\frac12}\frac{x_j - tx_i}{x_j - x_i}, \qquad \text{for } i,j\in\{1,\ldots,n\} \text{ with } i\ne j. \qquad \text{(cfnxdefn)}$$

If $w \in S_n$ then let $\mathrm{Inv}(w) = \{(i,j) \mid i,j\in\{1,\ldots,n\},\ i<j \text{ and } w(i)>w(j)\}$ and define

$$c_w(x) = \prod_{(i,j)\in\mathrm{Inv(w)}} c_{ij}(x). \qquad \text{(cfcnxw)}$$

With these notations the identity in Proposition 4.1 is

$$\mathbf{1}_0 = p_0 c_{w_0}(x^{-1}), \qquad \text{(symmopalt)}$$

where

$$c_{w_0}(x^{-1}) = \prod_{1\le i<j\le n} c_{ij}(x^{-1}) = \prod_{1\le i<j\le n} \frac{t^{-\frac12} - t^{\frac12}x_i^{-1}x_j}{1 - x_i^{-1}x_j} = t^{-\binom{n}{2}} \prod_{1\le i<j\le n} \frac{x_i - tx_j}{x_i - x_j}.$$

Proposition 4.2. *As operators on* $\mathbb{C}[x_1^{\pm1}, \ldots, x_n^{-1}]$,

$$\mathbf{1}_0 = p_0 c_{w_0}(x^{-1}).$$

Proof. Let $w\in S_n$. Using $T_i = s_ic_{i,i+1}(x^{-1}) + (t^{\frac12} - c_{i,i+1}(x))$ and a reduced word $w = s_{i_1}\cdots s_{i_\ell}$ and expanding, gives

$$T_w = T_{i_1}\cdots T_{i_\ell} = T_w = wc_w(x^{-1}) + \sum_{v<w} vb_v(x), \qquad \text{with } b_v(x)\in\mathbb{C}(x_1,\ldots,x_n).$$

Thus there are $a_v(x)\in\mathbb{C}(x_1,\ldots,x_n)$ such that

$$\mathbf{1}_0 = \sum_{w\in S_n} t^{-\frac12\ell(w_0w)}T_w = w_0c_{w_0}(x^{-1}) + \sum_{w<w_0} va_v(x), \qquad \text{(topterm)}$$

Since $p_0 = \sum_{w\in S_n} w$ then $s_ip_0 = p_0$ and

$$\begin{aligned} T_i(p_0c_{w_0}(x^{-1})) &= \big(c_{i,i+1}(x)s_i + (t^{\frac12} - c_{i,i+1}(x))\big)p_0c_{w_0}(x^{-1}) \\ &= \big(c_{i,i+1}(x) + (t^{\frac12} - c_{i,i+1}(x))\big)p_0c_{w_0}(x^{-1}) = t^{\frac12}\big(p_0c_{w_0}(x^{-1})\big). \end{aligned}$$

Since $\mathbf{1}_0$ is determined, up to multiplication by a constant, by the property that $T_i\mathbf{1}_0 = t^{\frac12}\mathbf{1}_0$ for $i\in\{1,\ldots,n-1\}$, it follows from (topterm) that, as operators on $\mathbb{C}[X]$,

$$\mathbf{1}_0 = p_0c_{w_0}(x^{-1}).$$

□

4.3.2 Symmetrizers and stabilizers

Let $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathbb{Z}^n$. Let

$$W_\lambda = \{w \in S_n \mid w\lambda = \lambda\} \qquad \text{which has longest element denoted } w_\lambda, \quad \text{and}$$
$$W^\lambda = \{\text{minimal length representatives of the cosets in } S_n/W_\lambda\},$$

so that W_λ is the stabilizer of λ under the action of S_n (acting by permutations of the coordinates). Let

$$p^\lambda = \sum_{u \in W^\lambda} u \qquad \text{and} \qquad p_\lambda = \sum_{v \in W_\lambda} v. \tag{partialWsymm}$$

The elements p^λ and p_λ have t-analogues given by

$$\mathbf{1}^\lambda = t^{-\frac{1}{2}\ell(w_0 w_\lambda)} \sum_{u \in W^\lambda} t^{\frac{1}{2}\ell(u)} T_u \qquad \text{and} \qquad \mathbf{1}_\lambda = t^{-\frac{1}{2}\ell(w_\lambda)} \sum_{v \in W_\lambda} t^{\frac{1}{2}\ell(v)} T_v. \tag{partialHsymm}$$

Then

$$p_0 = p^\lambda p_\lambda \qquad \text{and} \qquad \mathbf{1}_0 = \mathbf{1}^\lambda \mathbf{1}_\lambda.$$

The following proposition provides a formula for the bosonic symmetrizer $\mathbf{1}_0$ which is of striking utility (see the proof of the E-expansion and the proof that Macdonald's operators are the same as the elementary symmetric functions in the Y_i).

Proposition 4.3. *Let $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathbb{Z}^n$ and let w^λ be the longest element of the set W^λ. Use notations for the symmetrizers and c-functions as in (fullsymm), (partialWsymm), (partialHsymm) and (cfcnxw). Then, as operators on $\mathbb{C}[X]$,*

$$\mathbf{1}_0 = p^\lambda c_{w^\lambda}(x^{-1}) \mathbf{1}_\lambda.$$

Proposition 4.4. *Let $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathbb{Z}$ and let w^λ be the longest element of the set W^λ. Use notations for the symmetrizers and c-functions as in (fullsymm), (partialWsymm), (partialHsymm) and (cfcnxw). Then, as operators on $\mathbb{C}[X]$,*

$$\mathbf{1}_0 = p^\lambda c_{w^\lambda}(x^{-1}) \mathbf{1}_\lambda.$$

Proof. For $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$,

$$\mathrm{Inv}(w_\lambda) = \{(i,j) \mid i < j \text{ and } \lambda_i = \lambda_j \ \} \qquad \text{and} \qquad \mathrm{Inv}(w^\lambda) = \{(i,j) \mid i < j \text{ and } \lambda_i > \lambda_j \ \}.$$

If $u \in W_\lambda$ then $\lambda_{u(i)} > \lambda_{u(j)}$ if $\lambda_i > \lambda_j$ so that $u\mathrm{Inv}(w^\lambda) = \{(u(i), u(j)) \mid i < j \text{ and } \lambda_i > \lambda_j\} = \mathrm{Inv}(w^\lambda)$, which gives that $w_\lambda^{-1} c_{w^\lambda} = u c_{w^\lambda} = c^{w_\lambda}$ for $u \in W_\lambda$. This is the reason for the equalities

$$c_{w_0} = (w_\lambda^{-1} c_{w^\lambda}) c_{w_\lambda} = c_{w^\lambda} c_{w_\lambda} \qquad \text{and} \qquad p_\lambda c_{w^\lambda} = c_{w^\lambda} p_\lambda. \tag{cfcnsplit}$$

Replacing S_n by the group W_λ in the proof of Proposition 4.1 gives $\mathbf{1}_\lambda = p_\lambda c_{w_\lambda}(x^{-1})$. Using the relations in (cfcnsplit) and $\mathbf{1}_\lambda = p_\lambda c_{w_\lambda}(x^{-1})$ gives

$$\mathbf{1}_0 = p_0 c_{w_0}(x^{-1}) = p^\lambda p_\lambda c_{w_\lambda^{-1} w^\lambda}(x^{-1}) c_{w_\lambda}(x^{-1}) = p^\lambda c_{w^\lambda}(x^{-1}) p_\lambda c_{w_\lambda}(x^{-1}) = p^\lambda c_{w^\lambda}(x^{-1}) \mathbf{1}_\lambda.$$

$\square$

4.3.3 Symmetrizers and XY-parallelism

The *double affine Hecke algebra (of type GL_n)* is the algebra generated by symbols g and X_k and T_i for $i,k\in\mathbb{Z}$ with relations

$$T_{i+n}=T_i,\qquad X_{i+n}=q^{-1}X_i,\qquad X_kX_\ell=X_\ell X_k,\qquad \text{for } i,k,\ell\in\mathbb{Z};\qquad \text{(periodicityrelsF)}$$

$$T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1},\qquad T_iT_j=T_jT_i,\qquad T_i^2=(t^{\frac12}-t^{-\frac12})T_i+1,\qquad \text{(HeckerelsF)}$$

for $i,j\in\mathbb{Z}$ with $j\notin\{i-1,i+1\}$;

$$\begin{array}{l}T_ix_i=X_{i+1}T_i-(t^{\frac12}-t^{-\frac12})X_{i+1},\\ T_ix_{i+1}=X_iT_i+(t^{\frac12}-t^{-\frac12})X_{i+1},\end{array}\qquad X_{i+1}=T_iX_iT_i,\qquad\text{and}\qquad T_iX_j=X_jT_i,\quad \text{(XaffHeckerelsF)}$$

for $i\in\{1,\dots,n-1\}$ and $j\in\{1,\dots,n\}$ with $j\notin\{i,i+1\}$; and

$$gX_i=X_{i+1}g\qquad\text{and}\qquad gT_i=T_{i+1}g\qquad\text{for } i\in\mathbb{Z}.\qquad \text{(DAHArels2F)}$$

The *Cherednik-Dunkl operators* are $Y_1,\dots,Y_n$ given by

$$Y_1=gT_{n-1}\cdots T_1,\qquad\text{and}\qquad Y_{j+1}=T_j^{-1}Y_jT_j^{-1}\quad\text{for } j\in\{1,\dots,n-1\}.\qquad \text{(CDops)}$$

Define Y_i for $i\in\mathbb{Z}$ by setting

$$Y_{i+n}=q^{-1}Y_i\qquad\text{and let}\qquad g^\vee=x_1T_1\cdots T_{n-1}.$$

c-functions. For $i,j\in\mathbb{Z}$ with $i\neq j$ set

$$c_{ij}(X)=\frac{t^{-\frac12}-t^{\frac12}X_iX_j^{-1}}{1-X_iX_j^{-1}}\qquad\text{and}\qquad c_{ij}(Y)=\frac{t^{-\frac12}-t^{\frac12}Y_iY_j^{-1}}{1-Y_iY_j^{-1}}.\qquad \text{(cfnadefn)}$$

Y-intertwiners. For $i\in\{1,\dots,n-1\}$ define η_{s_i} by the equation

$$\eta_{s_i}=\frac{1}{c_{-\alpha_i^\vee}}(T_i^\vee+(c_{-\alpha_i^\vee}-t^{\frac12}))=\frac{1}{c_{-\alpha_i^\vee}}((T_i^\vee)^{-1}+(c_{-\alpha_i^\vee}-t^{-\frac12})).\qquad \text{(etaidefn)}$$

X-intertwiners. For $i\in\{1,\dots,n-1\}$ define ξ_{s_i} by the equation

$$\xi_{s_i}=\frac{1}{c_{-\alpha_i}}(T_i+(c_{-\alpha_i}-t^{\frac12}))=\frac{1}{c_{-\alpha_i}}((T_i)^{-1}+(c_{-\alpha_i}-t^{-\frac12}).\qquad \text{(xiidefn)}$$

If $w\in S_n$ and $w=s_{i_1}\cdots s_{i_\ell}$ is a reduced word for w define

$$\xi_w=\xi_{s_{i_1}}\cdots\xi_{s_{i_\ell}}\qquad\text{and}\qquad \eta_w=\eta_{s_{j_1}}\cdots\eta_{s_{j_m}},\qquad \text{(etawxiv)}$$

Define the

$$X\text{-symmetrizer}\quad p_0^X=\sum_{w\in W_0}\xi_w\ ,\qquad X\text{-antisymmetrizer}\quad e_0^X=\sum_{w\in W_0}\det(w_0w)\xi_w,$$
$$Y\text{-symmetrizer}\quad p_0^Y=\sum_{w\in W_0}\eta_w\ ,\qquad Y\text{-antisymmetrizer}\quad e_0^Y=\sum_{w\in W_0}\det(w_0w)\eta_w.$$

The *bosonic symmetrizer* and the *fermionic symmetrizer* are

$$\mathbf{1}_0=\sum_{z\in S_n}t^{\frac12(\ell(z)-\ell(w_0))}T_z\qquad\text{and}\qquad e_0=\sum_{w\in S_n}(-1)^{\ell(z)-\ell(w_0)}z.\qquad \text{(bosfersymm)}$$

The bosonic symmetrizer $\mathbf{1}_0$ is a t-analogue of p_0^X and p_0^Y and the fermionic symmetrizer ε_0 is a t-analogue of e_0^X and e_0^Y. The following Proposition rewrites the bosonic and fermionic symmetrizers in terms of the X-symmetrizers and the Y-symmetrizers. It is a reformulation of (symmopalt) which highlights the XY-parallelism in the DAHA.

Proposition 4.5. ([Mac03, (5.5.14) and (5.5.16)])

$$\mathbf{1}_0 = p_0^X c_{w_0}(X^{-1}) = p_0^Y c_{w_0}(Y) \qquad \textit{and} \qquad \varepsilon_0 = c_{w_0}(X) e_0^X = c_{w_0}(Y^{-1}) e_0^Y. \qquad \text{(slicksymmA)}$$

4.3.4 Symmetrizers, stabilizers and XY-parallelism

Let $\lambda \in (\mathfrak{a}_{\mathbb{Z}}^*)^+$. The stabilizer of λ under the action of W_0 is

$$W_\lambda = \{v \in W_0 \mid v\lambda = \lambda\} \qquad \text{and} \qquad w_\lambda \text{ denotes the longest element of } W_\lambda.$$

Let

$$W^\lambda \text{ be the set of minimal length representatives of the cosets in } W/W_\lambda.$$

Let w^λ be the longest element of W^λ so that $w_0 = w^\lambda w_\lambda$ with $\ell(w_0) = \ell(w^\lambda) + \ell(w_\lambda)$. Let

$$\begin{array}{llll}
p_X^\lambda = \sum_{u \in W^\lambda} \xi_u & \text{and} \quad p_\lambda^X = \sum_{v \in W_\lambda} \xi_v & \text{so that} & p_0^X = p_X^\lambda p_\lambda^X, \\
e_X^\lambda = \sum_{u \in W^\lambda} \det(w^\lambda u) \xi_u & \text{and} \quad e_\lambda^X = \sum_{v \in W_\lambda} \det(w_\lambda v) \xi_v & \text{so that} & e_0^X = e_X^\lambda e_\lambda^X, \\
p_Y^\lambda = \sum_{u \in W^\lambda} \eta_u & \text{and} \quad p_\lambda^Y = \sum_{v \in W_\lambda} \eta_v, & \text{so that} & p_0^Y = p_Y^\lambda p_\lambda^Y, \\
e_Y^\lambda = \sum_{u \in W^\lambda} \det(w^\lambda u) \eta_u & \text{and} \quad e_\lambda^Y = \sum_{v \in W_\lambda} \det(w_\lambda v) \eta_v & \text{so that} & e_0^Y = e_Y^\lambda e_\lambda^Y.
\end{array} \qquad \text{(pWs)}$$

The elements in (pWs) have t-analogues:

$$\begin{array}{ll}
\mathbf{1}^\lambda = t^{-\frac{1}{2}\ell(w^\lambda)} \sum_{u \in W^\lambda} (t^{\frac{1}{2}})^{\ell(u)} T_u & \text{and} \quad \mathbf{1}_\lambda = t^{-\frac{1}{2}\ell(w_\lambda)} \sum_{v \in W_\lambda} (t^{\frac{1}{2}})^{\ell(v)} T_v, \\
\varepsilon^\lambda = (-t^{-\frac{1}{2}})^{-\ell(w^\lambda)} \sum_{u \in W^\lambda} (-t^{-\frac{1}{2}})^{\ell(u)} T_u & \text{and} \quad \varepsilon_\lambda = (-t^{-\frac{1}{2}})^{-\ell(w_\lambda)} \sum_{v \in W_\lambda} (-t^{-\frac{1}{2}})^{\ell(v)} T_v.
\end{array} \qquad \text{(pHs)}$$

Then

$$\mathbf{1}_0 = \mathbf{1}^\lambda \mathbf{1}_\lambda \qquad \text{and} \qquad \varepsilon_0 = \varepsilon^\lambda \varepsilon_\lambda.$$

The following is a generalization of Proposition 4.5. It is a reformulation of (4.3) which highlights the XY-parallelism in the DAHA.

Proposition 4.6. *Let $\lambda \in (\mathfrak{a}_{\mathbb{Z}}^*)^+$ and let w^λ be the longest element of the set W^λ. Use notations of the symmetrizers and c-functions as in (cfnadefn), (bosfersymm), (pHs) and (pWs).*

$$\begin{array}{l}
\mathbf{1}_0 = p_X^\lambda c_{w^\lambda}(X^{-1}) \mathbf{1}_\lambda = p_Y^\lambda c_{w^\lambda}(Y) \mathbf{1}_\lambda \qquad \textit{and} \\
\varepsilon_0 = c_{w^\lambda}(X) e_X^\lambda \varepsilon_\lambda = c_{w^\lambda}(Y^{-1}) e_Y^\lambda \varepsilon_\lambda.
\end{array} \qquad \text{(symwparabA)}$$

4.4 Page 4: E-expansions

Proposition 4.7. *Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and let $S_n\lambda$ be the set of distinct rearrangements of λ. Then*

$$\begin{array}{l}
P_\lambda = \sum_{z \in W^\lambda} t^{\frac{1}{2}\ell(w^\lambda z)} \mathrm{ev}^\rho_{z\lambda}(c_{w^\lambda z}(Y)) E_{z\lambda} \qquad \textit{and} \\
A_{\lambda+\rho} = \sum_{z \in W_0} (-t^{\frac{1}{2}})^{\ell(w_0 z)} \mathrm{ev}^\rho_{z(\lambda+\rho)}(c_{w_0 z}(Y^{-1})) E_{z(\lambda+\rho)}.
\end{array}$$

Alternatively, letting $v_\mu \in S_n$ be the minimal length permutation such that $v_\mu\mu$ is weakly increasing,

$$P_\lambda = \sum_{\mu\in S_n\lambda} t^{\#\{i<j\ |\ \mu_i>\mu_j\}}\Big(\prod_{\substack{1\le i<j\le n\\ \mu_i>\mu_j}} \frac{1-q^{\mu_i-\mu_j}t^{v_\mu(j)-v_\mu(i)-1}}{1-q^{\mu_i-\mu_j}t^{v_\mu(j)-v_\mu(i)}}\Big)E_\mu \qquad \text{and}$$

$$A_{\lambda+\rho} = \sum_{\mu\in S_n(\lambda+\rho)} \Big(\Big(\prod_{\substack{1\le i<j\le n\\ \mu_i>\mu_j}} (-1)\Big(\frac{1-q^{\mu_i-\mu_j}t^{v_\mu(j)-v_\mu(i)+1}}{1-q^{\mu_i-\mu_j}t^{v_\mu(j)-v_\mu(i)}}\Big)\Big)E_\mu.$$

If $n=2$ and $m\in\mathbb{Z}_{>0}$ then

$$P_{(m,0)} = E_{(0,m)} + t^{\frac12}\mathrm{ev}^\rho_{(m,0)}(c_{12})E_{(m,0)} = E_{(0,m)} + t\Big(\frac{1-q^m}{1-q^mt}\Big)E_{(m,0)},$$

$$A_{m\omega_1} = E_{(0,m)} - t^{\frac12}\mathrm{ev}^\rho_{(m,0)}(c_{21})E_{(m,0)} = E_{-m\omega_1} - \frac{1-q^mt^2}{1-q^mt}E_{m\omega_1}.$$

To relate the expressions to the c-functions note that $t\Big(\frac{1-q^m}{1-q^mt}\Big) = t^{\frac12}\Big(\frac{t^{-\frac12}-t^{\frac12}q^{-m}t^{-1}}{1-q^{-m}t^{-1}}\Big)$. If $n=3$ then

$$\begin{aligned}P_{(2,1,0)} &= E_{(0,1,2)} + t\Big(\frac{1-q}{1-qt}\Big)E_{(1,0,2)} + t\Big(\frac{1-q}{1-qt}\Big)E_{(0,2,1)} + t^2\Big(\frac{1-qt}{1-qt^2}\Big)\Big(\frac{1-q^2}{1-q^2t}\Big)E_{(2,0,1)} \\ &\quad + t^2\Big(\frac{1-qt}{1-qt^2}\Big)\Big(\frac{1-q^2}{1-q^2t}\Big)E_{(1,2,0)} + t^3\Big(\frac{1-q}{1-qt}\Big)\Big(\frac{1-q^2t}{1-q^2t^2}\Big)\Big(\frac{1-q}{1-qt}\Big)E_{(2,1,0)}, \\ P_{(1,0,0)} &= E_{(0,0,1)} + t\Big(\frac{1-q}{1-qt}\Big)E_{(0,1,0)} + t^2\Big(\frac{1-q}{1-qt}\Big)\Big(\frac{1-qt}{1-qt^2}\Big)E_{(1,0,0)}\end{aligned}$$

For general n, if $\varepsilon_i = (0,\ldots,0,1,0,\ldots,0)$ is the sequence of length n with 1 in the ith spot and 0 elsewhere then

$$P_{(r,0,\ldots,0)} = \sum_{i=1}^n t^{n-i}\Big(\frac{1-q^r}{1-q^rt}\Big)\Big(\frac{1-q^rt}{1-q^rt^2}\Big)\cdots\Big(\frac{1-q^rt^{n-i-1}}{1-q^rt^{n-i}}\Big)E_{r\varepsilon_i} = \sum_{i=1}^n t^{n-i}\Big(\frac{1-q^r}{1-q^rt^{n-i}}\Big)E_{r\varepsilon_i}.$$

4.5 Page 5: Symmetrization of E_μ

The following Proposition shows that the symmetrization $\mathbf{1}_0E_\mu$ of the nonsymmetric Macdonald polynomial E_μ is always, up to an explicit constant factor, equal to the symmetric Macdonald polynomial P_λ.

Proposition 4.8. *Let $\mu=(\mu_1,\ldots,\mu_n)\in\mathbb{Z}^n$. Let $\lambda=(\lambda_1,\ldots,\lambda_n)$ be the weakly decreasing rearrangement of μ and let $z_\mu\in S_n$ be minimal length such that $\mu=z_\mu\lambda$. Let*

$$W_\lambda = \{y\in S_n\ |\ y\lambda=\lambda\} \qquad \text{and} \qquad W_\lambda(t) = \sum_{y\in W_\lambda} t^{\ell(y)}.$$

Then

$$P_\lambda = \frac{t^{\frac12\ell(w_0)}}{W_\lambda(t)}\Big(\frac{1}{t^{\frac12\ell(z_\mu)}\mathrm{ev}^\rho_\lambda(c_{z_\mu}(Y))}\Big)\mathbf{1}_0E_\mu.$$

Alternatively,

$$P_\lambda = \frac{t^{\frac12\ell(w_0)}}{W_\lambda(t)}\Big(\prod_{(i,j)\in\mathrm{Inv}(z_\mu)} \frac{1-q^{\lambda_i-\lambda_j}t^{j-i}}{1-q^{\lambda_i-\lambda_j}t^{j-i+1}}\Big)\mathbf{1}_0E_\mu.$$

4.6 Page 6: KZ families

For $\mu \in \mathbb{Z}^n$, let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ be the decreasing rearrangement of μ and let $z_\mu \in S_n$ be minimal length such that $\mu = z_\mu \lambda$. Define

$$f_\mu = E_\lambda^{z_\mu} = t^{\frac{1}{2}\ell(z_\mu)} T_{z_\mu} E_\lambda. \tag{4.7}$$

It follows from the identities in the last column of (CXlambdaaction) that

$$\{f_\mu \mid \mu \in S_n\lambda\} \quad \text{is another basis of } \mathbb{C}[X]^\lambda.$$

The following Proposition says that the $\{f_\mu \mid \mu \in \mathbb{Z}^n\}$ form a KZ-family, in the terminology of [KT06, Def. 3.3] (see also [CMW18, Def. 1.13], [CdGW15, (17), (18), (19)], [CdGW16, Def. 2]).

Proposition 4.9. *Let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n_{\geq 0}$. Let $i \in \{1, \ldots, n-1\}$ and let T_i and g be as defined in (8.1). Then*

$$t^{\frac{1}{2}} T_i f_\mu = \begin{cases} f_{s_i\mu}, & \text{if } \mu_i > \mu_{i+1}, \\ t f_\mu, & \text{if } \mu_i = \mu_{i+1}, \end{cases} \qquad \text{and} \qquad g f_\mu = q^{-\mu_n} f_{(\mu_n, \mu_1, \ldots, \mu_{n-1})}.$$

4.7 Lecture 4: Notes and references

Following [Fe11, Definition 4.4.2] and [Al16, Definition 5] and [Mac03, (5.7.6)] (Ferreira references private communication with Haglund), define the *permuted basement Macdonald polynomials* by

$$E_\mu^z = t^{-\frac{1}{2}\ell(w_0)} t^{\frac{1}{2}\ell(z)} T_z E_\mu, \qquad \text{for } \mu \in \mathbb{Z}^n \text{ and } z \in S_n. \tag{4.8}$$

For the *symmetrization of* E_μ see [Mac03, (5.7.1)] and [Mac95, Remarks after (6.8)]). See [Mac95, remarks after (6.8)] or [Mac03, (5.7.2)] for the explicit constant.

The fomulas for the symmetrizers $\mathbf{1}_0$ and ε_0 in Section 4.3 follow [Mac03, (5.5.14) and (5.5.16)].