1 Lecture 1, 23 February 2022: n-periodic permutations

1.1 The affine Weyl group

The (*type GL_n*) *finite Weyl group* is

$$W_{\mathrm{fin}} = S_n, \quad \text{the symmetric group of bijections } v\colon \{1,\ldots,n\} \to \{1,\ldots,n\}$$

with operation of composition of functions. The *type GL_n affine Weyl group W* is the group of *n-periodic permutations $w\colon \mathbb{Z}\to\mathbb{Z}$* i.e.,

$$\text{bijective functions } w\colon \mathbb{Z}\to\mathbb{Z} \text{ such that } w(i+n) = w(i)+n. \tag{1.1}$$

Any n-periodic permutation w is determined by its values $w(1),\ldots,w(n)$. Using $w(i+n)=w(i)+n$, any permutation $v\colon\{1,\ldots,n\}\to\{1,\ldots,n\}$ in S_n extends to an n-periodic permutation in W, and so $S_n \subseteq W$.

Define $\pi \in W$ by

$$\pi(i) = i+1, \quad \text{for } i \in \mathbb{Z}. \tag{1.2}$$

Define $s_0, s_1, \ldots, s_{n-1} \in W$ by

$$\begin{matrix} s_i(i) = i+1, \\ s_i(i+1) = i, \end{matrix} \qquad \text{and} \quad s_i(j) = j \ \text{ for } j \in \{0,1,\ldots,i-1,i+2,\ldots,n-1\}. \tag{1.3}$$

The finite Weyl group S_n is the subgroup of W generated by $s_1,\ldots,s_{n-1}$.

For $\mu = (\mu_1,\ldots,\mu_n)\in\mathbb{Z}^n$ define $t_\mu \in W$ by

$$t_\mu(1) = 1+n\mu_1, \quad t_\mu(2) = 2+n\mu_2, \quad \ldots, \quad t_\mu(n) = n+n\mu_n. \tag{1.4}$$

Then

$$W = \{t_\mu v \mid \mu\in\mathbb{Z}^n, v\in S_n\} \qquad \text{with} \qquad v t_\mu = t_{v\mu} v \ \text{ for } v \in S_n \text{ and } \mu\in\mathbb{Z}^n. \tag{1.5}$$

The map

$$\overline{}\colon W \to S_n \quad \text{given by} \qquad \overline{t_\mu v} = v, \quad \text{for } \mu\in\mathbb{Z}^n \text{ and } v \in S_n, \tag{1.6}$$

is a surjective group homomorphism.

1.2 Inversions

Let $w \in W$ be an n-periodic permutation. An *inversion of w* is

$$(j,k) \quad \text{with} \quad j<k \text{ and } w(j) > w(k).$$

If (j,k) is an inversion of w then $(j+\ell n, k+\ell n)$ is an inversion of w for $\ell\in\mathbb{Z}$ and so it is sensible to assume $j\in\{1,\ldots,n\}$ and define

$$\mathrm{Inv}(w) = \{(j,k) \mid j \in \{1,\ldots,n\},\ k\in\mathbb{Z},\ j<k \text{ and } w(j)>w(k)\}.$$

The number of elements of $\mathrm{Inv}(w)$,

$$\ell(w) = \#\mathrm{Inv}(w), \quad \text{is the } \textit{length of } w.$$

Proposition 1.1. *Let $\mu \in \mathbb{Z}^n$ and $v \in S_n$. Then*

$$\mathrm{Inv}(t_\mu v) = \Big(\bigcup_{\substack{i<j, v(i)<v(j) \\ \mu_{v(i)} \ge \mu_{v(j)}}} \bigcup_{\ell=0}^{\mu_j-\mu_i-1} \{(i, j+\ell n)\} \Big) \cup \Big(\bigcup_{\substack{i<j, v(i)>v(j) \\ \mu_{v(i)} \ge \mu_{v(j)}}} \bigcup_{\ell=0}^{\mu_j-\mu_i} \{(i, j+\ell n)\} \Big)$$
$$\cup \Big(\bigcup_{\substack{i<j, v(i)<v(j) \\ \mu_{v(i)} < \mu_{v(j)}}} \bigcup_{\ell=1}^{\mu_i-\mu_j} \{((j, i+\ell n)\} \Big) \cup \Big(\bigcup_{\substack{i<j, v(i)>v(j) \\ \mu_{v(i)} < \mu_{v(j)}}} \bigcup_{\ell=1}^{\mu_i-\mu_j-1} \{((j, i+\ell n)\} \Big)$$

For notational convenience when working with reduced words, let $s_\pi = \pi$. Then

$$\ell(s_\pi) = \ell(\pi) = 0 \quad \text{and} \quad \ell(s_i) = 1 \ \text{ for } i \in \{1, \dots, n-1\}.$$

Let $w \in W$. A *reduced word for* w is an expression of w as a product of $s_1, \dots, s_{n-1}$ and s_π,

$$w = s_{i_1} \dots s_{i_\ell} \quad \text{with } i_1, \dots, i_\ell \in \{1, \dots, n-1, \pi\} \quad \text{such that} \quad \ell(w) = \ell(s_{i_1}) + \dots + \ell(s_{i_\ell}).$$

1.3 The elements u_μ, v_μ, t_μ

As in (1.4), for $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$ define $t_\mu \in W$ by

$$t_\mu(1) = 1 + n\mu_1, \quad t_\mu(2) = 2 + n\mu_2, \quad \dots, \quad t_\mu(n) = n + n\mu_n.$$

Then

$$t_\mu = u_\mu v_\mu, \qquad \text{where } v_\mu \in S_n \text{ and } u_\mu \text{ is minimal length in the coset } t_\mu W_{\mathrm{fin}}. \tag{1.7}$$

Proposition 1.2. *Let $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n_{\ge 0}$. Let u_μ and v_μ be as defined in (1.7).*

(a) v_μ is the minimal length element of S_n such that $v_\mu \mu$ is (weakly) increasing.

(b) The permutation $v_\mu \colon \{1, \dots, n\} \to \{1, \dots, n\}$ is given by

$$v_\mu(i) = 1 + \#\{i' \in \{1, \dots, i-1\} \mid \mu_{i'} \le \mu_i\} + \#\{i' \in \{i+1, \dots, n\} \mid \mu_{i'} < \mu_i\}.$$

(c) The n-periodic permutations $u_\mu \colon \mathbb{Z} \to \mathbb{Z}$ and $u_\mu^{-1} \colon \mathbb{Z} \to \mathbb{Z}$ are given by

$$u_\mu(i) = v_\mu^{-1}(i) + n\mu_i \qquad \text{and} \qquad u_\mu^{-1}(i) = v_\mu(i) - n\mu_{v_\mu(i)} \qquad \text{for } i \in \{1, \dots, n\}.$$

(d) Let $|\mu_i - \mu_j|$ denote the absolute value of $\mu_i - \mu_j$. Then

$$\ell(t_\mu) = \sum_{\substack{i,j \in \{1,\dots,n\} \\ i<j}} |\mu_i - \mu_j|, \qquad \ell(v_\mu) = \#\{i < j \mid \mu_i > \mu_j\} \qquad \text{and} \qquad \ell(u_\mu) = \ell(t_\mu) - \ell(v_\mu).$$

Remark 1.3. Define an action of W on $\mathbb{Z}^n$ by

$$\begin{aligned} \pi(\mu_1, \dots, \mu_n) &= (\mu_n + 1, \mu_1, \dots, \mu_{n-1}) \qquad \text{and} \\ s_i(\mu_1, \dots, \mu_n) &= (\mu_1, \dots, \mu_{i-1}, \mu_{i+1}, \mu_i, \mu_{i+2}, \dots, \mu_n), \quad \text{for } i \in \{1, \dots, n\}. \end{aligned} \tag{1.8}$$

Then u_μ is the minimal length element of W such that $u_\mu(0, 0, \dots, 0) = (\mu_1, \dots, \mu_n)$. □

1.4 Boxes

Fix $n \in \mathbb{Z}_{>0}$. A *box* is an element of $\{1, \ldots, n\} \times \mathbb{Z}_{\geq 0}$ so that

$$\{\text{boxes}\} = \{(r, c) \mid r \in \{1, \ldots, n\},\ c \in \mathbb{Z}_{\geq 0}\}.$$

To conform to [Mac, p.2], we draw the box (r, c) as a square in row r and column c using the same coordinates as are usually used for matrices.

$$\text{The } \textit{cylindrical coordinate} \text{ of the box } (r, c) \text{ is the number } r + nc. \tag{1.9}$$

The *basement* is the set $\{(r, 0) \mid r \in \{1, \ldots, n\}\}$, so that the basement is the collection of boxes in the 0th column. Pictorially,

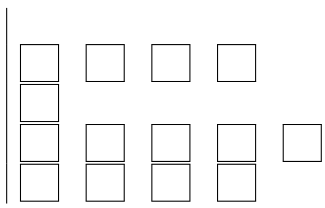
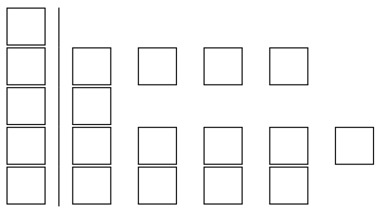
1 $(1,0)$	6 $(1,1)$	11 $(1,2)$	16 $(1,3)$	23 $(1,4)$	$\cdots$
2 $(2,0)$	7 $(2,1)$	12 $(2,2)$	17 $(2,3)$	22 $(2,4)$	$\cdots$
3 $(3,0)$	8 $(3,1)$	13 $(3,2)$	18 $(3,3)$	23 $(3,4)$	$\cdots$
4 $(4,0)$	9 $(4,1)$	14 $(4,2)$	19 $(4,3)$	24 $(4,4)$	$\cdots$
5 $(5,0)$	10 $(5,1)$	15 $(5,2)$	20 $(5,3)$	25 $(5,4)$	$\cdots$

with box (r, c) numbered $r+nc$.

Let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}_{\geq 0}^n$ an n-tuple of nonnegative integers. The *diagram of* μ is the set $dg(\mu)$ of boxes with μ_i boxes in row i and the *diagram of* μ *with basement* $\widehat{dg}(\mu)$ includes the extra boxes $(r, 0)$ for $r \in \{1, \ldots, n\}$:

$$\begin{aligned} dg(\mu) &= \{(r, c) \mid r \in \{1, \ldots, n\} \text{ and } c \in \{1, \ldots, \mu_r\}\} \quad \text{and} \\ \widehat{dg}(\mu) &= \{(r, c) \mid r \in \{1, \ldots, n\} \text{ and } c \in \{0, 1, \ldots, \mu_r\}\} \end{aligned}$$

It is often convenient to abuse notation and identify μ, $dg(\mu)$ and $\widehat{dg}(\mu)$ (because these are just different ways of viewing the sequence $(\mu_1, \ldots, \mu_n)$). For example, if $\mu = (0, 4, 1, 5, 4)$ then

$dg(\mu) =$  and $\widehat{dg}(\mu) =$  .

1.5 Affine coroots

Let $\mathfrak{a}_{\mathbb{Z}}$ be the set of $\mathbb{Z}$-linear combinations of symbols $\varepsilon_1^\vee, \ldots, \varepsilon_n^\vee, K$. The *affine coroots* are

$$\alpha^\vee_{i,j+\ell n} = \varepsilon_i^\vee - \varepsilon_j^\vee + \ell K \qquad \text{with } i, j \in \{1, \ldots, n\} \text{ and } i \neq j \text{ and } \ell \in \mathbb{Z}$$

(in the context of the corresponding affine Lie algebra the symbol K is the central element). The *shift* and *height* of an affine coroot are given by

$$\mathrm{sh}(\varepsilon_i^\vee - \varepsilon_j^\vee + \ell K) = -\ell \qquad \text{and} \qquad \mathrm{ht}(\varepsilon_i^\vee - \varepsilon_j^\vee + \ell K) = j - i. \tag{1.10}$$

The affine coroot corresponding to an inversion

$$(i, k) = (i, j + \ell n) \quad \text{with } i, j \in \{1, \ldots, n\} \text{ and } \ell \in \mathbb{Z}, \qquad \text{is} \quad \alpha^\vee_{i,j+\ell n} = \varepsilon_i^\vee - \varepsilon_j^\vee + \ell K. \tag{1.11}$$

Define a $\mathbb{Z}$-linear action of the affine Weyl group W on $\mathfrak{a}_\mathbb{Z}$ by

$$\pi^{-1}\varepsilon_1^\vee = \varepsilon_n^\vee + K, \quad \pi^{-1}\varepsilon_i^\vee = \varepsilon_{i-1}^\vee \ \text{ for } i \in \{2,\ldots,n\}, \tag{1.12}$$

$$s_i\varepsilon_i^\vee = \varepsilon_{i+1}^\vee, \quad s_i\varepsilon_{i+1}^\vee = \varepsilon_i^\vee, \quad s_i\varepsilon_j = \varepsilon_j^\vee \ \text{ if } j \in \{1,\ldots,n\} \text{ and } j \notin \{i,i+1\}.$$

If $\mu = (\mu_1,\ldots,\mu_n) \in \mathbb{Z}^n$ then $t_\mu\varepsilon_i^\vee = \varepsilon_i^\vee - \mu_i K$.

Let

$$\alpha_0^\vee = \alpha_{n,n+1}^\vee = \varepsilon_n^\vee - \varepsilon_1^\vee + K, \qquad \text{and} \qquad \alpha_i^\vee = \varepsilon_i^\vee - \varepsilon_{i+1}^\vee \ \text{ for } i \in \{1,\ldots,n-1\}.$$

Let $w \in W$ and let $w = s_{i_1}\cdots s_{i_\ell}$ be a reduced word for w. The *coroot sequence* of the reduced word $w = s_{i_1}\cdots s_{i_\ell}$ (recall that $s_\pi = \pi$) is

$$\text{the sequence } (\beta_k^\vee \mid k \in \{1,\ldots,\ell\} \text{ and } i_k \neq \pi\}) \text{ given by } \qquad \beta_k^\vee = s_{i_\ell}^{-1}\cdots s_{i_{k+1}}^{-1}\alpha_{i_k}^\vee. \tag{1.13}$$

Then, identifying inversions with affine coroots as in (1.11),

$$\mathrm{Inv}(w) = \{\beta_k^\vee \mid k \in \{1,\ldots,\ell\} \text{ and } k \neq \pi\} \tag{1.14}$$

(see [Mac03, (2.2.9)] or [Bou, Ch. VI §1 no. 6 Cor. 2]).

1.6 The box greedy reduced word for u_μ

Let $\mu \in \mathbb{Z}^n$.

$$\text{Write } (r,c) \in \mu \qquad \text{if } r \in \{1,\ldots,n\} \text{ and } c \in \mathbb{Z} \text{ with } c \le \mu_r.$$

For $(r,c) \in \mu$ define

$$u_\mu(r,c) = \#\{r' \in \{1,\ldots,r-1\} \mid \mu_{r'} < c \le \mu_r\} + \#\{r' \in \{r+1,\ldots,n\} \mid \mu_{r'} < c-1 < \mu_r\}.$$

The *box greedy reduced word for* u_μ is

$$u_\mu^\square = \prod_{(r,c)\in\mu} (s_{u_\mu(r,c)}\cdots s_1\pi), \tag{1.15}$$

where the product is over the boxes of μ *in increasing cylindrical wrapping order*. The following Proposition justifies the terminology box greedy *reduced word for* u_μ.

Proposition 1.4. *Let $\mu \in \mathbb{Z}^n$. For $r \in \{1,\ldots,n\}$ and $c \in \mathbb{Z}$ define*

$$v_\mu(r) = 1 + \#\{r' \in \{1,\ldots,r-1\} \mid \mu_{r'} \le \mu_r\} + \#\{r' \in \{r+1,\ldots,n\} \mid \mu_{r'} < \mu_r\}.$$

and

$$\mathrm{arm}_\mu(r,c) = \mu_r - c + 1.$$

The product $u_\mu^\square$ is a reduced word for u_μ, the inversion set of u_μ is

$$\mathrm{Inv}(u_\mu) = \bigcup_{(r,c)\in\mu} \bigcup_{i=1}^{u_\mu(r,c)} \{\varepsilon_{v_\mu(r)}^\vee - \varepsilon_i^\vee + \mathrm{arm}_\mu(r,c)K\} \qquad \textit{and} \qquad \ell(u_\mu) = \sum_{(r,c)\in\mu} u_\mu(r,c).$$

Remark 1.5. Let $\mu \in \mathbb{Z}^n_{\geq 0}$. For $(r, c) \in \mu$ define

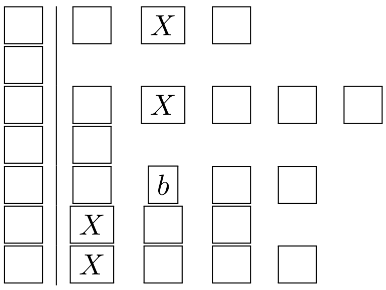
$$\mathrm{attack}_\mu(r, c) = \{(r', c) \in \mu \mid r' < r\} \sqcup \{(r', c-1) \in \mu \mid r' > r\}.$$

Then

$$u_\mu(r, c) = n - 1 - \#\mathrm{attack}_\mu(r, c).$$

For example, with $\mu = (3, 0, 5, 1, 4, 3, 4)$ and $b = (5, 2)$, which has cylindrical coordinate $b = 5+7\cdot 2 = 19$ the set $\mathrm{attack}_\mu(b)$ is pictured as

$\mathrm{attack}_\mu(b) =$



and $\qquad u_\mu(b) = u_\mu(5, 2) = 7 - 1 - 4 = 2.$

$\square$