10.05.2022
Lecture II ①
A. Ram

Koornwinder polynomials

Koornwinder polynomials are Macdonald polynomials for type CC_n. Up to now we have been treating type GL_n.

Classical types: Lie groups

Type A: $GL_n(\mathbb{C})$, $GL_n(\mathbb{R})$, $SL_n(\mathbb{C})$, $SL_n(\mathbb{R})$,
$PGL_n(\mathbb{C})$, $PGL_n(\mathbb{R})$, $U_n(\mathbb{C})$, etc...

Type B: $O_{2n+1}(\mathbb{C})$, $O_{2n+1}(\mathbb{R})$, $SO_{2n+1}(\mathbb{C})$, $SO_{2n+1}(\mathbb{R})$ etc

Type C: $Sp_{2n}(\mathbb{C})$, $Sp_{2n}(\mathbb{R})$, $PSp_{2n}(\mathbb{C})$ etc.

Type D: $O_{2n}(\mathbb{C})$, $O_{2n}(\mathbb{R})$, $SO_{2n}(\mathbb{C})$, $SO_{2n}(\mathbb{R})$, etc.

Classical types: finite reflection groups

Let $r, n, p \in \mathbb{Z}_{>0}$ with $\frac{r}{p} \in \mathbb{Z}$.

$$G(r,p,n) = \left\{ \begin{array}{l} w \in GL_n(\mathbb{C}) \text{ with exactly one nonzero} \\ \text{entry in each row and each column} \\ \text{(a) nonzero entries are } r^{th} \text{ roots of } 1 \\ \text{(b) } (\prod \text{nonzero entries})^{r/p} = 1. \end{array} \right\}$$

Type B: $W_{fin} = G(2,1,n)$

Type C: $W_{fin} = G(2,1,n)$

Type D: $G(2,2,n)$

Type A: $S_n = G(1,1,n) \subseteq GL_n(\mathbb{C})$

AND $S_n \subseteq GL_{n-1}(\mathbb{C})$ generated by

$$s_1 = \begin{array}{c|cccccc} & 1 & 2 & 3 & \cdots & n-1 \\ \hline 1 & -1 & 1 & & & \\ 2 & 0 & 1 & & & \\ 3 & & & 1 & & \\ \vdots & & & & \ddots & \\ n-1 & & & & & 1 \end{array} \quad \text{and} \quad s_i = \begin{array}{c|ccccccccc} & 1 & \cdots & i-2 & i-1 & i & i+1 & i+2 & \cdots & n-1 \\ \hline 1 & 1 & & & & & & & & \\ \vdots & & \ddots & & & & & & & \\ i-2 & & & 1 & & & & & & \\ i-1 & & & & 1 & 0 & 0 & & & \\ i & & & & 0 & -1 & 1 & & & \\ i+1 & & & & 0 & 0 & 1 & & & \\ i+2 & & & & & & & 1 & & \\ \vdots & & & & & & & & \ddots & \\ n-1 & & & & & & & & & 1 \end{array}$$

for $i \in \{2, \ldots, n-1\}$.

The Koornwinder polynomials are the bosonic Macdonald polynomials $P_\lambda(x_1, \ldots, x_n; q, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}}, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t)$ of type $CC_n = (C_n^\vee, C_n)$

The Askey-Wilson polynomials are the bosonic Macdonald polynomials $P_\lambda(x; q, t_1^{\frac{1}{2}}, u_1^{\frac{1}{2}}, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}})$ of type $CC_1 = (C_1^\vee, C_1)$. The correspondence to the original Askey-Wilson parameters is

$$a = q^{\frac{1}{2}} t_0^{\frac{1}{2}} u_0^{\frac{1}{2}}, \quad b = q^{\frac{1}{2}} t_0^{\frac{1}{2}} u_0^{-\frac{1}{2}}, \quad c = t_n^{\frac{1}{2}} u_n^{\frac{1}{2}}, \quad d = -t_n^{\frac{1}{2}} u_n^{\frac{1}{2}}$$

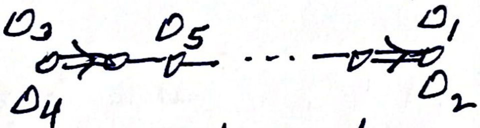
In the following table the left notation is that of Kac (for reduced cases only), the middle notation of Macdonald, the right is Bruhat-Tits.

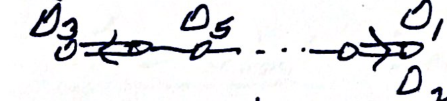
Kac = Macdonald = Bruhat-Tits.

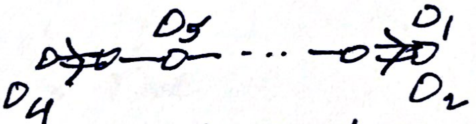
10.05.2022
Lecture 11
A. Ram

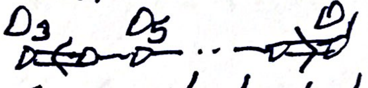
Macdonald polynomials of classical type

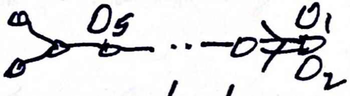
$$\begin{tikzcd}
{} & {(C_n^\vee, C_n) = C\text{-}BC_n^{II} \\ E_\mu(x; q, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}}, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t)} \arrow[dl, "t_0^{\frac{1}{2}} = u_0^{\frac{1}{2}}"'] \arrow[dr, "u_0^{\frac{1}{2}} = 1"] & {} & {} \\
{(C_n^\vee, BC_n) = C\text{-}BC_n^{I} \\ E_\mu(x; q, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}}, t_0^{\frac{1}{2}}, u_0^{\frac{1}{2}}, t)} \arrow[d, "t_n^{\frac{1}{2}} = u_n^{\frac{1}{2}}"] \arrow[dr, "u_0^{\frac{1}{2}} = 1"] & {} & {(BC_n, C_n) = C\text{-}BC_n^{IV} \\ E_\mu(x; q, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}}, t_0^{\frac{1}{2}}, 1, t)} \arrow[d, "t_n^{\frac{1}{2}} = u_n^{\frac{1}{2}}"'] \arrow[dr, "u_n^{\frac{1}{2}} = 1"] & {} \\
{D_{n+1}^{(2)} = C_n^\vee = C\text{-}B_n \\ E_\mu(x; q, t_n^{\frac{1}{2}}, t_n^{\frac{1}{2}}, t_0^{\frac{1}{2}}, t_0^{\frac{1}{2}}, t)} \arrow[d, "u_0^{\frac{1}{2}} = 1"'] & {(B_n, B_n^\vee) = B\text{-}BC_n \\ E_\mu(x; q, t_n^{\frac{1}{2}}, u_n^{\frac{1}{2}}, 1, 1, t)} \arrow[dl, "t_n^{\frac{1}{2}} = u_n^{\frac{1}{2}}"'] \arrow[dr, "u_n^{\frac{1}{2}} = 1"] & {A_{2n}^{(2)} = BC_n = C\text{-}BC_n^{III} \\ E_\mu(x; q, t_n^{\frac{1}{2}}, t_n^{\frac{1}{2}}, t_0^{\frac{1}{2}}, 1, t)} \arrow[dll, "t_0^{\frac{1}{2}} = u_0^{\frac{1}{2}}"] & {C_n^{(1)} = C_n = C_n \\ E_\mu(x; q, t_n^{\frac{1}{2}}, 1, t_0^{\frac{1}{2}}, 1, t)} \arrow[dl, "t_0^{\frac{1}{2}} = u_0^{\frac{1}{2}}"] \\
{B_n^{(1)} = B_n = B_n \\ E_\mu(x; q, t_n^{\frac{1}{2}}, t_n^{\frac{1}{2}}, 1, 1, t)} \arrow[dr, "u_n^{\frac{1}{2}} = 1"'] & {} & {A_{2n-1}^{(2)} = B_n^\vee = B\text{-}C_n \\ E_\mu(x; q, t_n^{\frac{1}{2}}, 1, 1, 1, t)} \arrow[dl, "t_n^{\frac{1}{2}} = u_n^{\frac{1}{2}}"] & {} \\
{} & {D_n^{(1)} = D_n = D_n \\ E_\mu(x; q, 1, 1, 1, 1, t)} & {} & {}
\end{tikzcd}$$

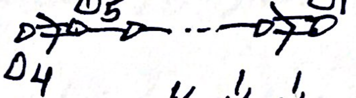


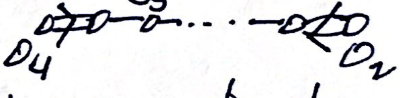


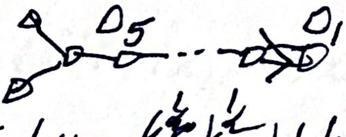


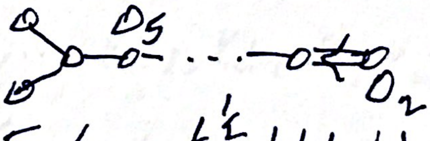


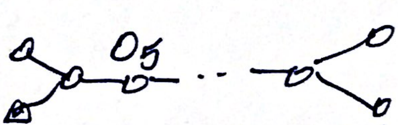












10.05.2022
Lecture 11
A. Ram

Comparing type GL_n and type CC_n

Parameters:

Type GL_n: $q, t^{\frac12} \in \mathbb{C}^\times$

Type CC_n: $q, t^{\frac12}, t_0^{\frac12}, u_0^{\frac12}, t_n^{\frac12}, u_n^{\frac12} \in \mathbb{C}^\times$

The finite Weyl group acts on $\mathbb{C}[x_1^{\pm1}, \ldots, x_n^{\pm1}]$

Type GL_n: The finite Weyl group is the symmetric group S_n generated by $s_1, \ldots, s_{n-1}$ with

$$(s_i f)(x_1, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_n)$$

for $i \in \{1, \ldots, n-1\}$ and $f \in \mathbb{C}[x_1^{\pm1}, \ldots, x_n^{\pm1}]$.

Type CC_n: The finite Weyl group W_{fin} is generated by $s_1, \ldots, s_{n-1}$ and s_n with

$$(s_n f)(x_1, \ldots, x_n) = f(x_1, \ldots, x_{n-1}, x_n^{-1})$$

$$(s_i f)(x_1, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_n)$$

for $i \in \{1, \ldots, n-1\}$ and $f \in \mathbb{C}[x_1^{\pm1}, \ldots, x_n^{\pm1}]$.

10.05.2022
Lecture 11 ⑤
A. Ram

Indexing of Macdonald polynomials

Type GL_n: $E_\mu(x_1,\ldots,x_n;q,t)\in\mathbb{C}[x_1^{\pm1},\ldots,x_n^{\pm1}]$ are indexed by $\mu=(\mu_1,\ldots,\mu_n)\in\mathbb{Z}^n$.

Type CC_n: $E_\mu(x_1,\ldots,x_n;q,t_0^{\frac12},u_0^{\frac12},t_n^{\frac12},u_n^{\frac12},t)\in\mathbb{C}[x_1^{\pm1},\ldots,x_n^{\pm1}]$ are indexed by $\mu=(\mu_1,\ldots,\mu_n)\in\mathbb{Z}^n$.

Type GL_n: The bosonic Macdonald polynomials $P_\lambda(x_1,\ldots,x_n;q,t)\in\mathbb{C}[x_1^{\pm1},\ldots,x_n^{\pm1}]^{S_n}$ are indexed by $\lambda=(\lambda_1,\ldots,\lambda_n)\in\mathbb{Z}^n$ with $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n$.

Type CC_n: The bosonic Macdonald polynomials $P_\lambda(x_1,\ldots,x_n;q,t,t_0^{\frac12},u_0^{\frac12},t_n^{\frac12},u_n^{\frac12})\in\mathbb{C}[x_1^{\pm1},\ldots,x_n^{\pm1}]^{W_{fin}}$ are indexed by

$$\lambda=(\lambda_1,\ldots,\lambda_n)\in\mathbb{Z}^n \text{ with } \lambda_1\geq\cdots\geq\lambda_n\geq0.$$

Action of H_Y on $\mathbb{C}[x_1^{\pm1},\ldots,x_n^{\pm1}]$

Type GL_n: Define operators $y_1,\ldots,y_n$ on $\mathbb{C}[x_1^{\pm1},\ldots,x_n^{\pm1}]$ by

$$(y_jf)(x_1,\ldots,x_n)=f(x_1,\ldots,x_{j-1},q^{-1}x_j,x_{j+1},\ldots,x_n)$$

for $j\in\{1,\ldots,n\}$.

The algebra H_Y is generated by T_π and $T_1, \ldots, T_{n-1}$ which act on $\mathbb{C}[x_1^{\pm1}, \ldots, x_n^{\pm1}]$ by

$$T_\pi = s_1 \cdots s_{n-1} y_n \quad \text{and} \quad T_i + t^{-\frac{1}{2}} = (1+s_i)\frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}} x_i^{-1} x_{i+1}}{1 - x_i^{-1} x_{i+1}}$$

for $i \in \{1, \ldots, n-1\}$.

Type CC_n: Define an operator s_0 on $\mathbb{C}[x_1^{\pm1}, \ldots, x_n^{\pm1}]$ by

$$(s_0 f)(x_1, \ldots, x_n) = f(q x_1^{-1}, x_2, \ldots, x_n)$$

for $f \in \mathbb{C}[x_1^{\pm1}, \ldots, x_n^{\pm1}]$. Define

$$c_{-\alpha_0}(x) = \frac{t_0^{-\frac{1}{2}}(1 - t_0^{\frac{1}{2}} u_0^{\frac{1}{2}} q^{-\frac{1}{2}} x_1)(1 - t_0^{\frac{1}{2}} u_0^{-\frac{1}{2}} q^{-\frac{1}{2}} x_1)}{(1 - q^{-\frac{1}{2}} x_1)^2}$$

$$c_{-\alpha_n}(x) = \frac{t_n^{-\frac{1}{2}}(1 - t_n^{\frac{1}{2}} u_n^{\frac{1}{2}} x_n^{-1})(1 - t_n^{\frac{1}{2}} u_n^{-\frac{1}{2}} x_n^{-1})}{(1 - x_n^{-1})^2}$$

$$c_{-\alpha_i}(x) = \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}} x_i^{-1} x_{i+1}}{1 - x_i^{-1} x_{i+1}} \quad \text{for } i \in \{1, \ldots, n-1\}.$$

The algebra H_Y is generated by T_0 and $T_1, \ldots, T_{n-1}$ and T_n which act on $\mathbb{C}[x_1^{\pm1}, \ldots, x_n^{\pm1}]$ by

$$T_0 + t_0^{-\frac{1}{2}} = (1+s_0) c_{-\alpha_0}(x), \quad T_i + t^{-\frac{1}{2}} = (1+s_i) c_{-\alpha_i}(x), \quad T_n + t_n^{-\frac{1}{2}} = (1+s_n) c_{-\alpha_n}(x)$$

for $i \in \{1, \ldots, n-1\}$.

10.05.2022
Lecture 11 ⑦
A. Ram

Cherednik-Dunkl operators

For type GL_n: The Cherednik-Dunkl operators are $Y_1, \dots, Y_n$ given by

$$Y_1 = T_\pi T_{n-1} \cdots T_1 \quad \text{and} \quad Y_j = T_{j-1}^{-1} Y_{j-1} T_{j-1}^{-1} \text{ for } j \in \{2, \dots, n\}.$$

Type CC_n: The Cherednik-Dunkl operators are $Y_1, \dots, Y_n$ given by

$$Y_1 = T_0 T_1 \cdots T_n \cdots T_1 \quad \text{and} \quad Y_j = T_{j-1}^{-1} Y_{j-1} T_{j-1}^{-1} \text{ for } j \in \{2, \dots, n\}.$$

Eigenvalues

Type GL_n: Let $v_\mu \in S_n$ be minimal length such that $v_\mu \mu$ is weakly increasing. Then

$$Y_i E_\mu = q^{-\mu_i} t^{-v_\mu(i)} t^{\frac{1}{2}(n+1)} E_\mu$$

Type CC_n: Let $v_\mu \in W_{fin}$ be minimal length such that $v_\mu \mu$ is weakly increasing with all entries ≤ 0. Then

$$Y_i E_\mu = q^{-\mu_i} t^{-v_\mu(i)} \left(t_0^{\frac{1}{2}} t_n^{\frac{1}{2}} t^{n}\right)^{\mathrm{sgn}(v_\mu(i))} E_\mu$$

Relations between X_j and $T_\pi^\vee$ and $T_0^\vee$

For $j\in\{1,\dots,n\}$, let X_j be the operator on $\mathbb{C}[x_1^{\pm1},\dots,x_n^{\pm1}]$ given by multiplication by x_j.

Type GL_n:

$$X_1 = T_\pi^\vee T_{n-1}^{-1}\cdots T_1^{-1} \quad\text{and}\quad X_j = T_{j-1}X_{j-1}T_{j-1}$$

for $j\in\{2,\dots,n\}$.

Type CC_n

$$X_1 = T_0^\vee T_1^{-1}\cdots T_n^{-1}\cdots T_1^{-1} \quad\text{and}\quad X_j = T_{j-1}X_{j-1}T_{j-1}$$

for $j\in\{2,\dots,n\}$.

Intertwiners

Type GL_n: The intertwiners $\tau_\pi^\vee$ and $\tau_1^\vee,\dots,\tau_{n-1}^\vee$ are given by

$$\tau_\pi^\vee = T_\pi^\vee \quad\text{and}\quad \tau_i^\vee + \frac{t^{-\frac12} - t^{\frac12}y_i y_{i+1}^{-1}}{1-y_i y_{i+1}^{-1}} = T_i + t^{-\frac12}$$

for $i\in\{1,\dots,n-1\}$. Then, for $j\in\{1,\dots,n\}$

$$\tau_\pi^\vee y_j = y_{j+1}\tau_\pi^\vee \quad\text{if } j\neq n, \qquad \tau_i^\vee y_i = y_{i+1}\tau_i^\vee$$

$$\tau_\pi^\vee y_n = q^{-1}y_1\tau_\pi^\vee, \qquad \tau_i^\vee y_{i+1} = y_i\tau_i^\vee$$

$$\tau_i^\vee y_j = y_j\tau_i^\vee \quad\text{if } j\notin\{i,i+1\}$$

for $i\in\{1,\dots,n-1\}$.

Type CC_n: Define

$$c_{\alpha_0^\vee}(Y) = \frac{t_0^{-\frac12}(1-t_0^{\frac12}u_0^{\frac12}q^{-\frac12}Y_1^{-1})(1-t_0^{\frac12}u_0^{-\frac12}q^{-\frac12}Y_1^{-1})}{(1-q^{-\frac12}Y_1^{-1})^2}$$

$$c_{\alpha_n^\vee}(Y) = \frac{t_n^{-\frac12}(1-t_n^{\frac12}u_n^{\frac12}Y_n)(1-t_n^{\frac12}u_n^{-\frac12}Y_n)}{(1-Y_n)^2} \quad \text{and}$$

$$c_{\alpha_i^\vee}(Y) = \frac{t^{-\frac12}-t^{\frac12}Y_iY_{i+1}^{-1}}{1-Y_iY_{i+1}^{-1}} \quad \text{for } i\in\{1,\dots,n-1\}.$$

The intertwiners $\tau_0^\vee$ and $\tau_1^\vee,\dots,\tau_{n-1}^\vee$ and $\tau_n^\vee$ are given by

$$\tau_0^\vee + c_{\alpha_0^\vee}(Y) = T_0 + t_0^{-\frac12}, \qquad \tau_n^\vee + c_{\alpha_n^\vee}(Y) = T_n + t_n^{-\frac12}$$

$$\tau_i^\vee + c_{\alpha_i^\vee}(Y) = T_i + t^{-\frac12}, \quad \text{for } i\in\{1,\dots,n-1\}.$$

Then, for $j\in\{1,\dots,n\}$

$$\tau_0^\vee Y_1 = q^{-1}Y_1^{-1}\tau_0^\vee, \qquad \tau_n^\vee Y_n = Y_n^{-1}\tau_n^\vee$$

$$\tau_0^\vee Y_j = Y_j\tau_0^\vee \text{ if } j\ne 1, \qquad \tau_n^\vee Y_j = Y_j\tau_n^\vee, \text{ if } j\ne n$$

and, for $i\in\{1,\dots,n-1\}$,

$$\tau_i^\vee Y_i = Y_{i+1}\tau_i^\vee,$$

$$\tau_i^\vee Y_{i+1} = Y_i\tau_i^\vee,$$

$$\tau_i^\vee Y_j = Y_j\tau_i^\vee \text{ if } j\notin\{i,i+1\}.$$