Monomial expansion of Macdonald polynomials

We wish to compute $a_\mu^\gamma, b_\lambda^\gamma, c_{\lambda+\rho}^\gamma$ where

$$E_\mu = \sum_\gamma a_\mu^\gamma x^\gamma \qquad \text{Electronic}$$

$$P_\lambda = \sum_\gamma b_\lambda^\gamma x^\gamma \qquad \text{Bosonic}$$

$$A_{\lambda+\rho} = \sum_\gamma c_{\lambda+\rho}^\gamma x^\gamma \qquad \text{Fermionic}$$

These are polynomials in $\mathbb{C}[X] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ which has basis

$$\{x^\gamma \mid \gamma \in \mathbb{Z}^n\} \quad \text{where } x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$$

if $\gamma = (\gamma_1, \ldots, \gamma_n)$.

__Theorem__ (Alcove walks formula)

$$E_\mu = \sum_F t^{\frac{1}{2}(\ell(f\,d(F)) - \ell(v_\mu^{-1}))} \left(\prod_{k\in F^-} q^{sh(k)} t^{ht(k)}\right) \left(\prod_{k\in F} \frac{t^{-\frac{1}{2}}(1-t)}{1-q^{sh(k)} t^{ht(k)}}\right) X^{\mathrm{end}}$$

__Creation formulas__

$$E_\mu = t^{-\frac{1}{2}\ell(v_\mu^{-1})} \tau^\vee_{u_\mu} 1$$

$$P_\lambda = \frac{t^{\frac{1}{2}\ell(w_0)}}{W_0(t)} \mathbf{1}_0 E_\lambda \quad \text{and} \quad A_{\lambda+\rho} = t^{\frac{1}{2}\ell(w_0)} \varepsilon_0 E_{\lambda+\rho}$$

The elements u_μ and v_μ in W

The group W of *n-periodic permutations* is the set of bijections $w: \mathbb{Z} \to \mathbb{Z}$ such that

$$w(i+n) = w(i) + n$$

with operation composition.

Define $s_1, \ldots, s_{n-1} \in W$ by

$s_i(i) = i+1$
$s_i(i+1) = i$ and $s_i(j) = j$ for $j \in \{1, \ldots, n\}$ with $j \notin \{i, i+1\}$.

Define $s_\pi \in W$ by $s_\pi(i) = i+1$.

For $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n_{\geq 0}$ define

$$u_\mu = \prod_{(r,c) \in \mu} s_{u_\mu(r,c)} \cdots s_2 s_1 s_\pi$$

where the product is in increasing order by the values $r + nc$ and

$$u_\mu(r,c) = \#\{r' < r \mid (r',c) \notin \mu\} + \#\{r' > r \mid (r', c-1) \notin \mu\}$$

(write $(r,c) \in \mu$ if $r \in \{1, \ldots, n\}$ and $c \in \{1, \ldots, \mu_r\}$).

Define $v_\mu \in S_n$ by

$$v_\mu(r) = \#\{r' > r \mid \mu_{r'} < \mu_r\} + \#\{r' < r \mid \mu_{r'} \leq \mu_r\}.$$

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Alcove walks

Let $w_k = s_{i_{k+1}} \cdots s_{i_\ell}$ where $u_\mu = s_{i_1} \cdots s_{i_\ell}$

Define $a_k, b_k \in \{1, \ldots, n\}$ and $c_k, d_k \in \mathbb{Z}$ by

$$w_k^{-1}(i_k) = a_k + c_k n \quad \text{and} \quad w_k^{-1}(i_k+1) = b_k + d_k n.$$

Define

$$sh(k) = d_k - c_k \quad \text{and} \quad ht(k) = b_k - a_k$$

Let $F \subseteq \{1, \ldots, \ell\}$ such that if $s_{i_k} = s_\pi$ then $k \notin F$.

The alcove walk of F is the sequence of elements of W

$$aw(F) = (z_0, z_1, \ldots, z_\ell) \quad \text{given by}$$

$$z_0 = 1 \quad \text{and} \quad z_k = \begin{cases} z_{k-1}, & \text{if } k \in F, \\ z_{k-1} s_{i_k}, & \text{if } k \notin F \end{cases}$$

Define $\gamma^{(k)} = (\gamma_1^{(k)}, \ldots, \gamma_n^{(k)}) \in \mathbb{Z}^n$ and $v_k \in S_n$ by

$$z_k(i) = v_k(i) + n\gamma^{(k)}_{v_k(i)} \quad \text{for } i \in \{1, \ldots, n\}$$

Define

$$end(F) = \gamma^{(\ell)}, \quad fd(F) = v_\ell \quad \text{and}$$

$$F^- = \{k \in F \mid \ell(v_k s_{i_k}) < \ell(v_k)\}.$$

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Operator form of the alcove walks formula

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Let

$$\tau^\vee_{u_\mu} = \prod_{(v,c)\in\mu} \tau^\vee_{u_\mu(v,c)} \cdots \tau^\vee_2 \tau^\vee_1 \tau^\vee_\pi$$

Theorem In the double affine Hecke algebra

$$\tau^\vee_{u_\mu} = \sum_F X^{\mathrm{end}(F)} T_{\mathrm{fd}(F)} \left(\prod_{k\in F^-} q^{\mathrm{sh}(k)} y_{a_k} y^{-1}_{b_k}\right)\left(\prod_{k\in F} \frac{t^{\frac12}(1-t)}{1-q^{\mathrm{sh}(k)} y_{a_k} y^{-1}_{b_k}}\right)$$

Then

$$E_\mu = t^{-\frac12 \ell(v^{-1}_\mu)} \tau^\vee_{u_\mu}\cdot 1$$

$$= t^{-\frac12\ell(v^{-1}_\mu)} \sum_F X^{\mathrm{end}(F)} T_{\mathrm{fd}(F)} \left(\prod_{k\in F^-} q^{\mathrm{sh}(k)} t^{\mathrm{ht}(k)}\right)\left(\prod_{k\in F} \frac{t^{\frac12}(1-t)}{1-q^{\mathrm{sh}(k)} t^{\mathrm{ht}(k)}}\right)$$

since $y_a y_b^{-1}\cdot 1 = t^{b-a}$ Then use $T_w\cdot 1 = t^{\frac12 \ell(w)}$ to get

$$E_\mu = \sum_F t^{\frac12(\ell(\mathrm{fd}(F))-\ell(v^{-1}_\mu)+|F|)} q^{\sum_{k\in F^-}\mathrm{sh}(k)} t^{\sum_{k\in F^-}\mathrm{ht}(k)} \left(\prod_{k\in F} \frac{1-t}{1-q^{\mathrm{sh}(k)} t^{\mathrm{ht}(k)}}\right) X^{\mathrm{end}(F)}$$

So

$$a^\gamma_\mu = \sum_{\substack{F\\ \mathrm{end}(F)=\gamma}} t^{\frac12(\ell(\mathrm{fd}(F))-\ell(v^{-1}_\mu)-|F|)} q^{\sum_{k\in F^-}\mathrm{sh}(k)} t^{\sum_{k\in F^-}\mathrm{ht}(k)} \left(\prod_{k\in F^-} \frac{1-t}{1-q^{\mathrm{sh}(k)} t^{\mathrm{ht}(k)}}\right)$$

Example illustrating how to execute the proof of the theorem

$$t_2^\vee t_1^\vee t_\pi^\vee = \left(T_2 + \frac{t^{-1/2}(1-t)}{1-Y_2^{-1}Y_3}\right) t_1^\vee t_\pi^\vee$$

$$= (T_2 + f_{32}^+) t_1^\vee t_\pi^\vee$$

$$= T_2 t_1^\vee t_\pi^\vee + t_1^\vee f_{31}^+ t_\pi^\vee = T_2 t_1^\vee t_\pi^\vee + t_1^\vee t_\pi^\vee f_{20}^+$$

$$= T_2(T_1 + f_{21}^+) t_\pi^\vee + (T_1 + f_{21}^+) t_\pi^\vee f_{20}^+$$

$$= T_2 T_1 t_\pi^\vee + T_2 t_\pi^\vee f_{10}^+ + T_1 t_\pi^\vee f_{20}^+ + t_\pi^\vee f_{10}^+ f_{20}^+$$

$$= x_3 + x_1 T_2 T_1 T_2 f_{10}^+ + x_2 T_2 f_{20}^+ + x_1 T_1 T_2 f_{10}^+ f_{20}^+$$

since

$$x_1 = t_\pi^\vee T_2^{-1} T_1^{-1}, \quad x_2 = T_1 t_\pi^\vee T_2^{-1}, \quad x_3 = T_2 T_1 t_\pi^\vee$$

so

$$t_2^\vee t_1^\vee t_\pi^\vee = x_3 + x_1 T_{s_2 s_1 s_2} \frac{t^{-1/2}(1-t)}{1-Y_1 Y_0^{-1}} + x_2 T_2 \frac{t^{-1/2}(1-t)}{1-Y_2 Y_0^{-1}}$$

$$+ x_1 T_{s_1 s_2} \frac{t^{-1/2}(1-t)^2}{(1-Y_1 Y_0^{-1})(1-Y_2 Y_0^{-1})}$$

$$= x_3 + x_1 T_{s_2 s_1 s_2} \frac{t^{-1/2}(1-t)}{1-q Y_1 Y_3^{-1}} + x_2 T_{s_2} \frac{t^{-1/2}(1-t)}{1-q Y_2 Y_3^{-1}}$$

$$+ x_1 T_{s_1 s_2} \frac{t^{-1/2}(1-t)^2}{(1-q Y_1 Y_3^{-1})(1-q Y_2 Y_3^{-1})}.$$

Column strict tableaux

Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}_{\geq 0}^n$ with $\lambda_1 \geq \cdots \geq \lambda_n$.

A <u>column strict tableau</u> of shape λ is a filling T of the boxes of λ such that

(a) column entries strictly increase top to bottom

(b) row entries weakly increase left to right.

$$T = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 \\ \hline 2 & 3 & 3 & 3 & 4 \\ \hline 3 & 4 & 5 & 5 \\ \cline{1-4} 5 \\ \cline{1-1} \end{array}$$

$T: \text{boxes} \to \{1, \ldots, n\}$

$b \mapsto T(b) = $ value in box b.

Let

$$x^T = x_1^{\#1\text{'s in }T} x_2^{\#2\text{'s in }T} \cdots x_n^{\#n\text{'s in }T}$$

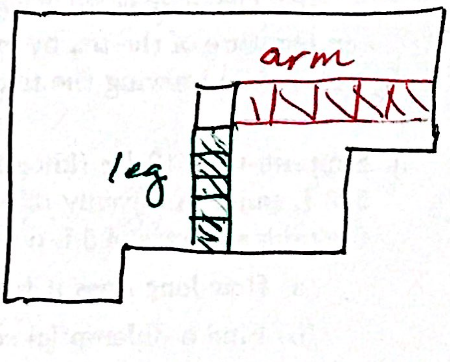
<u>Theorem</u> (Formula for the Schur function)

$$s_\lambda = \sum_T x^T$$

For $(r, c) \in \lambda$ set

$\text{arm}_\lambda(r,c) = \{(r, c') \in \lambda \mid c' > c\}$

$\text{leg}_\lambda(r,c) = \{(r', c) \in \lambda \mid r' > r\}$



Formula for P_λ

Let $i \in \{1, \ldots, n\}$ with $i > T(b)$

Define

$$a(b, \le i) = \{ b' \in arm_\lambda(b) \mid T(b') \le i \}$$
$$l(b, \le i) = \{ b' \in leg_\lambda(b) \mid T(b') \le i \}$$

$$a(b, < i) = \{ b' \in arm_\lambda(b) \mid T(b') < i \}$$
$$l(b, < i) = \{ b' \in leg_\lambda(b) \mid T(b') < i \}$$

and

$$h_T(b, \le i) = \frac{1 - q^{a(b,\le i)} t^{l(b, \le i)+1}}{1 - q^{a(b,\le i)+1} t^{l(b,\le i)}}$$

$$h_T(b, < i) = \frac{1 - q^{a(b,< i)} t^{l(b, < i)+1}}{1 - q^{a(b,< i)+1} t^{l(b,< i)}}$$

Theorem (Formula for P_λ)

$$P_\lambda = \sum_T \psi_T x^T,$$

where

$$\psi_T = \prod_{b \in \lambda} \psi_T(b)$$

and

$$\psi_T(b) = \prod_{\substack{i > T(b) \\ i \in T(arm_\lambda(b)) \\ i \notin T(leg_\lambda(b))}} \frac{h_T(b, < i)}{h_T(b, \le i)}.$$