Macdonald polynomials are eigenvectors

Let $n \in \mathbb{Z}_{>0}$ and $q, t^{\frac{1}{2}} \in \mathbb{C}^{\times}$.

The double affine Hecke algebra $\tilde{H}$ has

generators x_k for $k \in \mathbb{Z}$, T_i for $k \in \mathbb{Z}$ and $T_\pi = g$

relations $x_{i+n} = q^{-1} x_i$, $T_{i+n} = T_i$

$x_k x_l = x_l x_k$, $T_i^2 = (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) T_i + 1$,

$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$, $T_i T_j = T_j T_i$ if $j \notin \{i-1, i+1\}$

$g x_i = x_{i+1} g$, $g T_i = T_{i+1} g$

$x_{i+1} = T_i x_i T_i$ and $T_i x_j = x_j T_i$ if $j \notin \{i, i+1\}$

$T_i x_i = x_{i+1} T_i - (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) x_{i+1}$

$T_i x_{i+1} = x_i T_i + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) x_{i+1}$

Cherednik-Dunkl operators $y_1, \dots, y_n$ given by

$y_1 = g T_{n-1} \cdots T_1$ and $y_{j+1} = T_j^{-1} y_j T_j^{-1}$

Theorem $y_i y_j = y_j y_i$.

Action of DAHA on $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$

Define operators on $\mathbb{C}[X] = \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ by

$$(s_i f)(x_1, \dots, x_n) = f(x_1, \dots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \dots, x_n)$$

$$(y_j f)(x_1, \dots, x_n) = f(x_1, \dots, x_{j-1}, q^{-1} x_j, x_{j+1}, \dots, x_n)$$

and

$$\partial_i = \frac{1}{x_i - x_{i+1}} (1 - s_i)$$

for $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, n\}$.

<u>Theorem</u> An action of $\tilde{H}$ on $\mathbb{C}[X]$ is given by

x_j acts by multiplication by x_j

$$T_i = t^{\frac{1}{2}} x_{i+1} \partial_i - t^{-\frac{1}{2}} \partial_i x_{i+1}$$

$$g = s_1 \cdots s_{n-1} y_n$$

for $i \in \{1, \dots, n-1\}$ and $j \in \{1, \dots, n\}$.

For $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$, let $v_\mu \in S_n$ be minimal length such that $v_\mu \mu$ is weakly increasing,

$$v_\mu(r) = \#\{r' < r \mid \mu_{r'} \leq \mu_r\} + \#\{r' > r \mid \mu_{r'} < \mu_r\}.$$

<u>Theorem</u> For $j \in \{1, \dots, n\}$ and $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$,

$$y_j E_\mu = q^{-\mu_j} t^{-(v_\mu(j) - 1) + \frac{1}{2}(n-1)} E_\mu$$

Intertwiners

Define

$$\tau_\pi^\vee = x_1 T_1 \cdots T_{n-1} \quad \text{and} \quad \tau_i^\vee = T_i + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - y_i^{-1} y_{i+1}}$$

for $i \in \{1, \ldots, n-1\}$. Then let

$$y_{j+n} = q^{-1} y_j \quad \text{for } j \in \mathbb{Z}.$$

Proposition For $i \in \{1, \ldots, n-1\}$,

$$\tau_i^\vee y_i = y_{i+1} \tau_i^\vee$$
$$\tau_i^\vee y_{i+1} = y_i \tau_i^\vee \quad \text{and} \quad \tau_i^\vee y_j = y_j \tau_i^\vee \text{ for } j \in \{1, \ldots, n\} \text{ with } j \notin \{i, i+1\}$$

and $\tau_\pi^\vee y_j = y_{j+1} \tau_\pi^\vee$ for $j \in \mathbb{Z}$.

Let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}_{\geq 0}^n$. Write $(r, c) \in \mu$ if $r \in \{1, \ldots, n\}$ and $c \in \{1, \ldots, \mu_r\}$

Define

$$u_\mu(r, c) = \#\{r' < r \mid (r', c) \notin \mu\} + \#\{r' > r \mid (r', c-1) \notin \mu\}$$

Theorem

$$E_\mu = \Big(\prod_{(r,c) \in \mu} \tau^\vee_{u_\mu(r,c)} \cdots \tau_2^\vee \tau_1^\vee \tau_\pi^\vee \Big) \cdot 1$$

where the product is in increasing order by the values $r + nc$.

The polynomial representation as an induced module

H_Y is the subalgebra of $\tilde{H}$ generated by

$T_1, \dots, T_{n-1}$ and T_H.

Define a 1-dim'l H_Y-module triv = span$\{\mathbb{1}_Y\}$ with

$$(*) \quad T_i \mathbb{1}_Y = t^{1/2} \mathbb{1}_Y \quad \text{and} \quad T_H \mathbb{1}_Y = t^{0/2} \mathbb{1}_Y$$

Since $H_Y \to \tilde{H}$ then

$$\tilde{H}\text{-modules} \xrightarrow{\mathrm{Res}^{\tilde{H}}_{H_Y}} H_Y\text{-modules}$$
$$M \longmapsto \mathrm{Res}^{\tilde{H}}_{H_Y}(M)$$

Pullback or Lifting

Induction

$$H_Y\text{-modules} \xrightarrow{\mathrm{Ind}^{\tilde{H}}_{H_Y}} \tilde{H}\text{-modules}$$
$$N \longmapsto \mathrm{Ind}^{\tilde{H}}_{H_Y}(N)$$

Is a left adjoint of $\mathrm{Res}^{\tilde{H}}_{H_Y}$, i.e. it is determined by

$$\mathrm{Hom}(\mathrm{Ind}(N), M) = \mathrm{Hom}(N, \mathrm{Res}(M))$$

$$\begin{tikzcd} N \arrow[r] \arrow[dr] & \mathrm{Ind}(N) \arrow[d, dashed] \\ & M \end{tikzcd}$$

Universal Property

This is the same as saying

$\mathrm{Ind}_{H_Y}^{\tilde{H}}(\mathrm{triv})$ is the $\tilde{H}$-module

<u>generated by triv with relations (*)</u>

So

$$\mathrm{Ind}_{H_Y}^{\tilde{H}}(\mathrm{triv}) = \tilde{H}\mathbb{1}_Y = \mathrm{span}\{X^\mu \mathbb{1}_Y \mid \mu \in \mathbb{Z}^n\}$$

where $X^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}$ if $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$.

$$\mathbb{C}[X] \xrightarrow{\sim} \tilde{H}\mathbb{1}_Y$$
$$X^\mu \longmapsto X^\mu \mathbb{1}_Y \quad \text{as } \tilde{H}\text{-modules.}$$

<u>XY-parallelism</u>

$$X_{i+1} = T_i X_i T_i \quad \text{and} \quad Y_{j+1}^{-1} = T_j Y_j^{-1} T_j$$

Let

$$g^\vee = X_1 T_1 \cdots T_{n-1}$$

H_Y is the subalgebra generated by

$T_1, \dots, T_{n-1}$ and g

H_X is the subalgebra generated by

$T_1, \dots, T_{n-1}$ and $g^\vee$

H_0 is the subalgebra generated by

$T_1, \dots, T_{n-1}$.

Then

$$\tilde{H} = \mathbb{C}[X] \otimes H_Y$$
$$= H_X \otimes \mathbb{C}[Y]$$
$$= \mathbb{C}[X] \otimes H_0 \otimes \mathbb{C}[Y]$$

as vector spaces. In other words

H_0 has basis $\{T_z \mid z \in S_n\}$

H_X has basis $\{X^\mu T_z \mid \mu \in \mathbb{Z}^n, z \in S_n\}$

H_Y has basis $\{T_z Y^{\lambda^\vee} \mid \lambda^\vee \in \mathbb{Z}^n, z \in S_n\}$

$\tilde{H}$ has basis $\{X^\mu T_z Y^{\lambda^\vee} \mid \mu \in \mathbb{Z}^n, \lambda^\vee \in \mathbb{Z}^n, z \in S_n\}$

where

$$Y^{\lambda^\vee} = y_1^{\lambda_1} \cdots y_n^{\lambda_n} \quad \text{if} \quad \lambda^\vee = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$$

$$X^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n} \quad \text{if} \quad \mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$$

and

$$T_z = T_{i_1} \cdots T_{i_\ell} \quad \text{if} \quad z = s_{i_1} \cdots s_{i_\ell}$$

with $\{i_1, \dots, i_\ell\} \in \{1, \dots, n-1\}$ and $\ell = \ell(z)$.