1.7 The exponential and the logarithm

The **exponential function** is the element e^x of $\mathbb{Q}[[x]]$ given by

$$e^x = \sum_{k\in\mathbb{Z}_{>0}} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

Theorem 1.2. *As an element of* $\mathbb{Q}[[x,y]$ *(which has* $xy = yx$*),*

$$e^{x+y} = e^x e^y.$$

HW: Show that $e^0 = 1$.
HW: Show that $e^{-x} = \frac{1}{e^x}$.

The **logarithm** is

$$\log(1+x) = \sum_{k\in\mathbb{Z}_{>0}} (-1)^{k-1}\frac{x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$$

Theorem 1.3. *Let*

$$G = \{p(x) \in \mathbb{Q}[[x]] \mid p(0) = 1\} \qquad \textit{and} \qquad \mathfrak{g} = \{p(x) \in \mathbb{Q}[[x]] \mid p(0) = 0\}$$

Then

(a) $\log(1 + (e^x - 1)) = e^{\log(1+x)} - 1 = x.$

(b) G *is a commutative group under multiplication,* $\mathfrak{g}$ *is a commutative group under addition and*

$$\begin{array}{ccc} G & \longrightarrow & \mathfrak{g} \\ p & \longmapsto & e^p - 1 \end{array} \qquad \textit{is bijective and satisfies} \quad \varphi(p_1p_2) = \varphi(p_1) + \varphi(p_2).$$