5.9 Limits and multiplication: proof

Theorem 5.5. (Limits and multiplication)
Let $n \in \mathbb{Z}_{>0}$. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ and $g\colon \mathbb{R}^n \to \mathbb{R}$ be functions and let $a \in \mathbb{R}^n$.

$$\textit{Assume that} \quad \lim_{x\to a} f(x) \quad \textit{and} \quad \lim_{x\to a} g(x) \qquad \textit{exist.}$$

Then $\lim_{x\to a}(f(x)g(x)) = \big(\lim_{x\to a} f(x)\big)\big(\lim_{x\to a} g(x)\big)$.

Proof.

Let $l_1 = \lim_{x\to a} f(x)$ and $l_2 = \lim_{x\to a} g(x)$.
To show: $\lim_{x\to a}(f(x)g(x)) = l_1 l_2$.
To show: If $e \in \mathbb{Z}_{>0}$ then there exists $d \in \mathbb{Z}_{>0}$ such that

$$\text{if } x \in \mathbb{R}^n \text{ is within } 10^{-d} \text{ of } a \text{ then } f(x)g(x) \text{ is within } 10^{-e} \text{ of } l_1 l_2.$$

Assume $e \in \mathbb{Z}_{>0}$.
Let $r, s \in \mathbb{Z}_{>0}$ such that $|\ell_1| < 10^r$ and $|\ell_2| < 10^s$.
Since $\lim_{x\to a} f(x) = l_1$ then we know that there exists $d_1 \in \mathbb{Z}_{>0}$ such that

$$\text{if } x \in \mathbb{R}^n \text{ is within } 10^{-d_1} \text{ of } a \text{ and } f(x) \text{ is within } 10^{-(e+s+1)} \text{ of } l_1.$$

Since $\lim_{x\to a} f(x) = l_2$ then we know that there exists $d_2 \in \mathbb{Z}_{>0}$ such that

$$\text{if } x \in \mathbb{R}^n \text{ is within } 10^{-d_2} \text{ of } a \text{ and } f(x) \text{ is within } 10^{-(e+r+1)} \text{ of } l_2.$$

Let $d = \max(d_1, d_2)$.
Assume $x \in \mathbb{R}^n$ is within 10^{-d} of a.
To show: $f(x)g(x)$ is within 10^{-e} of $l_1 l_2$.

$$\begin{aligned}
|f(x)g(x) - l_1 l_2| &= |(f(x) - l_1)g(x) + l_1(g(x) - l_2)| \\
&\le |(f(x) - l_1)g(x)| + |l_1(g(x) - l_2)|, \quad \text{by the triangle inequality,} \\
&= |(f(x) - l_1)(g(x) - l_2) + (f(x) - l_1)l_2| + |l_1|\,|g(x) - l_2| \\
&\le |f(x) - l_1)(g(x) - l_2)| + |f(x) - l_1)l_2| + |l_1|\,|g(x) - l_2| \\
&\le |f(x) - l_1|\,|g(x) - l_2| + |f(x) - l_1|\,|l_2| + |l_1|\,|g(x) - l_2| \\
&\le |f(x) - l_1|\,|g(x) - l_2| + |f(x) - l_1|\,10^s + 10^r\,|g(x) - l_2| \\
&\le 10^{-(e+r+1)} \cdot 10^{-(e+s+1)} + 10^{-(e+s+1)}10^s + 10^r 10^{-(e+r+1)} \\
&= 10^{-e}(10^{-(e+r+s+2)} + 10^{-1} + 10^{-1}) < 10^{-e} \cdot 1 = 10^{-e}.
\end{aligned}$$

So $f(x)g(x)$ is within 10^{-e} of $l_1 l_2$.
So there exists $d \in \mathbb{Z}_{>0}$ such that

$$\text{if } x \in \mathbb{R}^n \text{ is within } 10^{-d} \text{ of } a \text{ then } f(x)g(x) \text{ is within } 10^{-e} \text{ of } l_1 l_2.$$

So $\lim_{x\to a}(f(x)g(x)) = l_1 l_2$.

□