1.15 Limits and addition, scalar multiplication, multiplication, composition and order

The *tolerance set* is

$$\mathbb{E} = \{10^{-1}, 10^{-2}, \ldots\}.$$

Let $m, n \in \mathbb{Z}_{>0}$ and let $f\colon \mathbb{R}^m \to \mathbb{R}^n$. Let $a \in \mathbb{R}^m$ and $\ell \in \mathbb{R}^n$.

$$\lim_{x\to a} f(x) = \ell \qquad \text{means}$$

$$\text{if } \varepsilon \in \mathbb{E} \text{ then there exists } \delta \in \mathbb{E} \text{ such that} \quad \text{if } d(x,a) < \delta \text{ then } d(f(x),\ell) < \varepsilon.$$

Let $a_1, a_2, \ldots$ be a sequence in $\mathbb{R}^m$. Let $\ell \in \mathbb{R}^m$.

$$\lim_{n\to\infty} a_n = \ell \qquad \text{means}$$

$$\text{if } \varepsilon \in \mathbb{E} \text{ then there exists } N \in \mathbb{Z}_{>0} \text{ such that} \quad \text{if } n \in \mathbb{Z}_{\geq N} \text{ then } d(a_n,\ell) < \varepsilon.$$

Theorem 1.5. *Let $n \in \mathbb{Z}_{>0}$. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ and $g\colon \mathbb{R}^n \to \mathbb{R}$ be functions and let $a \in X$.*

$$\textit{Assume that} \quad \lim_{x\to a} f(x) \quad \textit{and} \quad \lim_{x\to a} g(x) \qquad \textit{exist.}$$

Then

(a) $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$,

(b) If $c \in \mathbb{R}$ then $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$,

(c) $\lim_{x\to a}(f(x)g(x)) = \big(\lim_{x\to a} f(x)\big)\big(\lim_{x\to a} g(x)\big)$.

Theorem 1.6. *Let $m, n, p \in \mathbb{Z}_{>0}$ Let $f\colon \mathbb{R}^n \to \mathbb{R}^p$ and $g\colon \mathbb{R}^m \to \mathbb{R}^n$ be functions and let $a \in \mathbb{R}^m$ and $\ell \in \mathbb{R}^n$.*

$$\textit{Assume that} \quad \lim_{x\to a} g(x) \ \textit{and} \ \lim_{x\to a} f(g(x)) \ \textit{exist} \qquad \textit{and} \qquad \lim_{x\to a} g(x) = \ell.$$

Then

$$\lim_{y\to \ell} f(y) = \lim_{x\to a} f(g(x)).$$

Theorem 1.7.

(a) Let $(a_1, a_2, \ldots)$ and $(b_1, b_2, \ldots)$ be sequences in $\mathbb{R}$.
Assume that $\lim_{n\to\infty} a_n$ *and* $\lim_{n\to\infty} b_n$ *exist and*

$$\textit{if } n \in \mathbb{Z}_{>0} \textit{ then } a_n \leq b_n. \qquad \textit{Then} \qquad \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n.$$

(b) Let $n \in \mathbb{Z}_{>0}$ and let $f\colon \mathbb{R}^n \to \mathbb{R}$ and $g\colon \mathbb{R}^n \to \mathbb{R}$ be functions. Let $a \in \mathbb{R}^n$.
Assume that $\lim_{x\to a} f(x)$ *and* $\lim_{x\to a} g(x)$ *exist and*

$$\textit{if } x \in X \textit{ then } f(x) \leq g(x). \qquad \textit{Then} \qquad \lim_{x\to a} f(x) \leq \lim_{x\to a} g(x).$$