1.18.4 Harmonic series and the Riemann zeta function

Let $s \in \mathbb{C}$. The *Riemann zeta function at s* is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The *harmonic series* is $\zeta(1)$. A *p-series* is $\zeta(p)$ for $p \in \mathbb{R}_{>0}$.

Theorem 1.15. *Assume $p \in \mathbb{R}_{>0}$. Then*

$$\zeta(p) \text{ converges if and only if } p \in \mathbb{R}_{>1}.$$

Proof. Case 1: $p = 1$. In this case $\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges since

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}} + \cdots > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots.$$

Case 2: $p \in \mathbb{R}_{<1}$. Then $\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges since

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots > 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots.$$

Case 3: $p \in \mathbb{R}_{>1}$. Then $\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}$ converges since

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^p} &= 1 + \underbrace{\frac{1}{2^p} + \frac{1}{3^p}} + \underbrace{\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}} + \cdots \\
&< 1 + \frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \cdots \\
&= 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \frac{1}{8^{p-1}} + \cdots \\
&= 1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2 + \left(\frac{1}{2^{p-1}}\right)^3 + \cdots \\
&= \frac{1}{1 - \frac{1}{2^{p-1}}} = \frac{2^{p-1}}{2^{p-1} - 1}.
\end{aligned}$$

□