1.11 Functions

Functions are for comparing sets.

Let S and T be sets. A *function from S to T* is a subset $\Gamma_f \subseteq S \times T$ such that

$$\text{if } s \in S \quad \text{then} \quad \text{there exists a unique } t \in T \text{ such that } (s, t) \in \Gamma_f.$$

Write

$$\Gamma_f = \{(s, f(s)) \mid s \in S\}$$

so that the function Γ_f can be expressed as

$$\text{an "assignment"} \qquad \begin{array}{rccc} f\colon & S & \to & T \\ & s & \mapsto & f(s) \end{array}$$

which must satisfy

(a) If $s \in S$ then $f(s) \in T$, and

(b) If $s_1, s_2 \in S$ and $s_1 = s_2$ then $f(s_1) = f(s_2)$.

Let S and T be sets.

- Two functions $f\colon S \to T$ and $g\colon S \to T$ are *equal* if they satisfy

$$\text{if } s \in S \quad \text{then} \quad f(s) = g(s).$$

- A function $f\colon S \to T$ is *injective* if f satisfies the condition

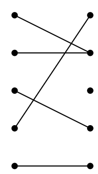
$$\text{if } s_1, s_2 \in S \text{ and } f(s_1) = f(s_2) \quad \text{then} \quad s_1 = s_2.$$

- A function $f\colon S \to T$ is *surjective* if f satisfies the condition

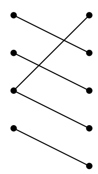
$$\text{if } t \in T \quad \text{then} \quad \text{there exists } s \in S \text{ such that } f(s) = t.$$

- A function $f\colon S \to T$ is *bijective* if f is both injective and surjective.

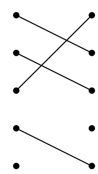
Examples. It is useful to visualize a function $f\colon S \to T$ as a graph with edges $(s, f(s))$ connecting elements $s \in S$ and $f(s) \in T$. With this in mind the following are examples:



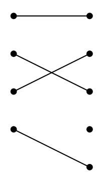
(a) a function



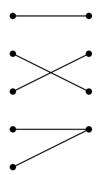
(b) not a function



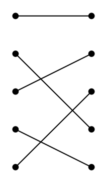
(c) not a function



(d) an injective function



(e) a surjective function



(f) a bijective function

In these pictures the elements of the left column are the elements of the set S and the elements of the right column are the elements of the set T. In order to be a function the graph must have exactly one edge adjacent to each point in S. The function is injective if there is at most one edge adjacent to each point in T. The function is surjective if there is at least one edge adjacent to each point in T.