1.19 Derivatives

The addition and multiplication on $\mathbb{R}$ is what makes the set $\mathcal{O}_{\mathbb{R}} = \{f\colon \mathbb{R} \to \mathbb{R}\}$ form an $\mathbb{R}$-algebra with addition, scalar multiplication and multiplication given by

$$(f+g)(x) = f(x) + g(x), \quad (cf)(x) = cf(x), \quad (fg)(x) = f(x)g(x),$$

for $f, g \in \mathcal{O}_{\mathbb{R}}$ and $c \in \mathbb{R}$.
A *derivative with respect to* x on $\mathcal{O}_{\mathbb{R}}$ is the function $\frac{d}{dx}\colon \mathcal{O}_{\mathbb{R}} \to \mathcal{O}_{\mathbb{R}}$ such that

$$\frac{d(f+g)}{dx} = \frac{df}{dx} + \frac{dg}{dx}, \quad \frac{d(cf)}{dx} = c\frac{df}{dx}, \quad \frac{d(fg)}{dx} = f\frac{dg}{dx} + \frac{df}{dx}g \quad \text{and} \quad \frac{dx}{dx} = 1,$$

for $f, g \in \mathcal{O}_{\mathbb{R}}$ and $c \in \mathbb{R}$ and where x denotes the identity function $\mathrm{id}\colon \mathbb{R} \to \mathbb{R}$.

Theorem 1.16. *(Chain rule and power formula)*

$$\frac{d(f \circ g)}{dx} = \frac{df}{dg}\frac{dg}{dx} \qquad \textit{and} \qquad \frac{d(f^g)}{dx} = f^g\Big(\frac{g}{f}\frac{df}{dx} + \log f\frac{dg}{dx}\Big).$$

Theorem 1.17. *If* $n \in \mathbb{Z}_{\geq 0}$ *then*

$$\frac{dx^n}{dx} = nx^{n-1} \qquad \textit{and} \qquad \frac{d\ e^x}{dx} = e^x.$$

Theorem 1.18. *If* $a \in \mathbb{C}$ *then*

$$\frac{d\ x^a}{dx} = ax^{a-1}, \qquad \frac{d\log x}{dx} = \frac{1}{x}, \qquad \frac{d\sin x}{dx} = \cos x, \qquad \frac{d\ \cos x}{dx} = -\sin x.$$

1.20 Integrals

The *integral* is backwards of the derivative,

$$\text{if} \quad \frac{dg}{dx} = f \qquad \text{then} \qquad \int f dx = g,$$

so that

$$\int \frac{df}{dx}\, dx = f, \quad \text{up to a constant.}$$

The product rule gives the formula for *integration by parts*:

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx \qquad (\text{since } \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}).$$

The chain rule gives the formula for *substitution*:

$$\int u dv = \int u\frac{dv}{dx}dx \qquad (\text{since } \frac{du}{dx} = \frac{du}{dv}\frac{dv}{dx}).$$