GTLA Lecture 08.09.2020

Let G and H be groups.
The direct product of G and H

is
$$G \times H = \{(g,h) \mid g \in G, h \in H\}$$

$(\mathbb{R}^2 = \{(x,y) \mid x \in \mathbb{R}, y \in \mathbb{R}\})$.

with
$$(g_1,h_1)\cdot(g_2,h_2) = (g_1 * g_2, h_1 h_2)$$

Example $\mu_2 = \{1,-1\}$ and

$\mu_2 \times \mu_2 = \{(1,1),(1,-1),(-1,1),(-1,-1)\}$

The multiplication table

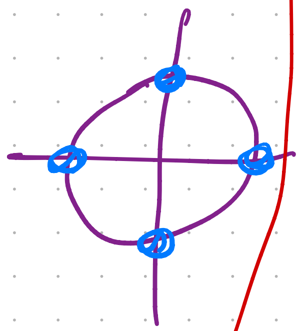
$\circ$	$(1,1)$	$(1,-1)$	$(-1,1)$	$(-1,-1)$
$(1,1)$	$(1,1)$	$(1,-1)$	$(-1,1)$	$(-1,-1)$
$(1,-1)$	$(1,-1)$	$(1,1)$	$(-1,-1)$	$(-1,1)$
$(-1,1)$	$(-1,1)$	$(-1,-1)$	$(1,1)$	$(1,-1)$
$(-1,-1)$	$(-1,-1)$	$(-1,1)$	$(1,-1)$	$(1,1)$

The orders of the elements:

order$(1,1) = 1$, order$(-1,1) = 2$

order$(1,-1) = 2$, order$(-1,-1) = 2$.

"Klein 4-group".



compare

Example $\mu_4 = \{1, i, -1, -i\}$

$= \{1, i, i^2, i^3\}$

So

$\mu_4 \cong \mathbb{Z}/4\mathbb{Z} = \{0, 1, 2, 3\}$

In μ_4,

order$(1) = 1$, order$(-1) = 2$

order$(i) = 4$, order$(-i) = 4$

Card$(\mu_2 \times \mu_2) = 4 =$ Card(μ_4)

but the orders of the elements are <u>not</u> the same.

So $\mu_2 \times \mu_2 \not\cong \mu_4$

Let G and H be groups.

A <u>homormorphism from G to H</u> is a function $f: G \to H$ such that

(a) If $g_1, g_2 \in G$ then

$$f(g_1 g_2) = f(g_1) \heartsuit f(g_2).$$

(b) $f(1) = ☺$

(c) If $g \in G$ then $f(g^{-1}) = (b)f$

Let $f: G \to H$ be a homomorphism.

$$\ker f = \{ g \in G \mid f(g) = ☺ \}.$$

$$\operatorname{im} f = \{ f(g) \mid g \in G \}.$$

A <u>normal subgroup</u> of G is a subgroup K such that if $n \in K$ and $g \in G$ then $g^{-1} n g \in K$.

(conjugates of elements in K are still in K)

<u>Proposition</u> Let $f: G \to H$ be a homomorphism (morphism in the category of groups).

(a) $\ker f$ is a normal subgroup of G.

(b) $\mathrm{im} f$ is a subgroup of H

<u>Proof</u> (a) To show:

(aa) If $n_1, n_2 \in \ker f$ then $n_1 n_2 \in \ker f$.

(ab) $1 \in \ker f$

(ac) If $n \in \ker f$ then $n^{-1} \in \ker f$

(ad) If $n \in \ker f$ and $g \in G$ then $g^{-1} n g \in \ker f$.

(aa) Assume ~~$n_1, n_2 \in \ker f$.~~ $n_1 \in \ker f$ and $n_2 \in \ker f$.

To show: $n_1 n_2 \in \ker f$

To show: $f(n_1 n_2) = ☺$

Since $n_1, n_2 \in \ker f$ then

$f(n_1) = ☺$ and $f(n_2) = ☺$

So $f(n_1 n_2) = f(n_1) ♡ f(n_2)$

= ☺ ♡ ☺ = ☺

So $n_1 n_2 \in \ker f$.

(ab) To show: $1 \in \ker f$.

To show: $f(1) =$ ☺

By condition (b) in the definition of a homomorphism $f(1) =$ ☺.

So $1 \in \ker f$.

(ac) To show: If $n_1 \in \ker f$ then $n_1^{-1} \in \ker f$.

Assume $n_1 \in \ker f$.

To show: $n_1^{-1} \in \ker f$.

Since $n_1 \in \ker f$ then $f(n_1) =$ ☺

To show: $f(n_1^{-1}) =$ ☺

By condition (c) in the defn. of homom.

$f(n_1^{-1}) = f(n_1)^{-1}$

= ☺$^{-1}$ = ☺

So $n_1^{-1} \in \ker f$.

(ad) To show: If $n \in \ker f$ and $g \in G$ then $g^{-1}ng \in \ker f$.

Assume $n \in \ker f$ and $g \in G$.

To show: $g^{-1}ng \in \ker f$.

To show: $f(g^{-1}ng) = \text{☺}$

Since $n \in \ker f$ then $f(n) = \text{☺}$

Then

$$\begin{aligned} f(g^{-1}ng) &= f(g^{-1}) \heartsuit f(n) \heartsuit f(g) \\ &= f(g^{-1}) \heartsuit \underbrace{\text{☺} \heartsuit f(g)} \\ &= f(g^{-1}) \heartsuit f(g) \\ &= f(g)^{-1} \heartsuit f(g) \\ &= \text{☺} \end{aligned}$$

where the first equality is by condition (a) in defn. of homom., the 3rd equality is identity in a group, and 4th equality is condition (c) in definition of homom.

So $\ker f$ is a normal subgroup of G.

(b) To show: $\mathrm{im} f$ is a subgroup of H.

(go back to ·, / for H).

To show: (ba) If $h_1, h_2 \in \mathrm{im} f$ then $h_1 h_2 \in \mathrm{im} f$.

(bb) $1 \in \mathrm{im} f$

(bc) If $h \in \mathrm{im} f$ then $h^{-1} \in \mathrm{im} f$.

(ba) Assume $h_1, h_2 \in \mathrm{im} f$.

To show: $h_1 h_2 \in \mathrm{im} f$

To show: There exists $g \in G$ such that $f(g) = h_1 h_2$

Since $h_1 \in \mathrm{im} f$ then ~~there~~ there exists $g_1 \in G$ such that $f(g_1) = h_1$

Since $h_2 \in \mathrm{im} f$ then there exists $g_2 \in G$ such that $f(g_2) = h_2$.

Let $g = g_1 g_2$.

To show: $f(g) = h_1 h_2$.

$$f(g) = f(g_1 g_2) = f(g_1) f(g_2)$$
$$= h_1 h_2.$$

(bb) To show: $1 \in \operatorname{im} f$.

To show: There exists $g \in G$ such that $f(g) = 1$.

Let $g = 1$.

To show: $f(g) = 1$.

$$f(g) = f(1) = 1.$$

(bc) To show: If $h \in \operatorname{im} f$ then $h^{-1} \in \operatorname{im} f$.

Assume $h \in \operatorname{im} f$.

Then there exists $g \in G$ such that $f(g) = h$.

To show: $h^{-1} \in \operatorname{im} f$.

Since

$$f(g^{-1}) = f(g)^{-1} = h^{-1}$$

then $h^{-1} \in \operatorname{im} f$. ∎

$$G \times G \to G$$
$$(a,b) \mapsto ab$$