Tutorial 5

Main topics: Cayley-Hamilton theorem, Jordan normal form

1. Find the Jordan normal form for each of the following matrices using their characteristic and minimal polynomials.

$$A = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix} \qquad c(X) = m(X) = (X+1)^2$$

$$B = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix} \qquad c(X) = m(X) = (X-2)^2(X-3)$$

$$C = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix} \qquad c(X) = (X-2)^2(X-3), m(X) = (X-2)(X-3)$$

2. A complex matrix has characteristic polynomial $(X-2)^2(X-5)^3$. Describe all possibilities for its Jordan normal form (up to re-ordering of the Jordan blocks). Write down the minimal polynomial in each case.

3. Let A be a 5×5 complex matrix. Find the possible Jordan normal forms of A if:
   (a) A has characteristic polynomial $(X-\alpha)^5$ and $\operatorname{rank}(A-\alpha I) = 2$.
   (b) A has characteristic polynomial $(X-2)^2(X-5)^3$, the 2-eigenspace has dimension 1 and the 5-eigenspace has dimension 2.

4. Which of the following pairs of matrices (over $\mathbb{C}$) are similar?
   (a) $\begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$
   (b) $\begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 5 \\ 0 & -1 \end{bmatrix}$
   (c) $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

   [Hints: (i) similar matrices have the same trace, determinant and eigenvalues. (ii) Two matrices over the complex numbers are similar if and only if they have the same Jordan normal form (up to re-ordering on Jordan blocks).]

5. (a) Find two distinct matrices $A, B \in M_7(\mathbb{C})$ that are in Jordan normal form with unique eigenvalue 0, and have the same minimal polynomial, characteristic polynomial and dimension of 0-eigenspace.
   (b) Compute the dimensions of the subspaces V_i of $V = \mathbb{C}^7$ defined by

$$V_i = \{v \in V \mid A^i v = 0\}.$$

   Do the same for B, and compare the results.