Tutorial 11

Main topics: group actions, orbit-stabiliser theorem, Sylow theorems

1. Let G be a group. Show that the following gives an action of G on $X = G$:

$$g \cdot x = xg^{-1} \text{ for } g \in G, x \in X$$

2. Let $G = \{e, a, b, ab\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ act as the symmetries of a rectangle, with a and b as shown below.

What is the stabiliser and orbit of: (a) a vertex (b) the midpoint of an edge?

3. Let $\mathrm{GL}(2,\mathbb{R})$ act on $\mathbb{R}^2$ in the usual way: $A \cdot x = Ax$ for $A \in \mathrm{GL}(2,\mathbb{R})$ and $x \in M_{2\times 1}(\mathbb{R})$.
Describe the stabiliser and orbit of: (a) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

4. A group G of order 9 acts on a set X having 16 elements. Show that there must be at least one point in X that is fixed by all elements of G.

5. Find the conjugacy class and centraliser of:

   (a) $(1\,2) \in S_3$ (b) $(1\,2\,3) \in S_3$

   Check that |conjugacy class| × |centraliser = $|S_3|$ in each case.

6. Let G be a group of order $84 = 2^2 \times 3 \times 7$. What can you say about the number of

   (a) Sylow 2-subgroups? (b) Sylow 3-subgroups? (c) Sylow 7-subgroups?

   Explain why G must have a normal subgroup of order 7.

7. Let G be an abelian group of order n. Prove that G has a unique Sylow p-subgroup for each prime $p \mid n$.

8. Let G be a group of order $30 = 2 \times 3 \times 5$, and let n_p denote the number of Sylow p-subgroups of G.
   (a) Prove that $n_3 = 1$ or $n_5 = 1$. Hence G must have a normal subgroup of order 3 or 5.
   (b) Prove that if $n_2 = 15$ then $n_3 = n_5 = 1$.