MAST20022 Group Theory and Linear Algebra

Sample exam 5

Question A1.

1. Let $a(X), b(X) \in \mathbb{R}[X]$. Give the definition of a *greatest common divisor* of $a(X)$ and $b(X)$.

2. Use the Euclidean algorithm to find the greatest common divisor (in $\mathbb{N}$) of 836 and 494.

Question A2.

1. For each of the following, find the multiplicative inverse in $\mathbb{Z}/110\mathbb{Z}$ or explain why no inverse exists:

   a) $[12]_{110}$ b) $[57]_{110}$

2. Find the smallest $m \in \mathbb{N}$ such that the remainder when m is divided by 7 is 4 and the remainder when m is divided by 22 is 14.

Question A3.
Let $f : (\mathbb{F}_5)^3 \to (\mathbb{F}_5)^3$ be the linear transformation given by

$$f([x]_5, [y]_5, [z]_5) = ([3x - z]_5, [-x + 2y + z]_5, [x + z]_5)$$

1. Calculate the eigenvalues of f.

2. Find the minimal polynomial of f.

Question A4.

(a) Let V be a vector space over $\mathbb{C}$. Give the definition of an inner product on V.

(b) Show that $\langle A, B\rangle = \operatorname{tr}((\overline{B})^t A)$ defines an inner product on $M_2(\mathbb{C})$.

Question A5.

Let $f : V \to V$ be a linear transformation of a finite dimensional vector space V. Show that there are bases $\mathcal{B}$ and $\mathcal{C}$ of V such that $[f]_{\mathcal{C},\mathcal{B}}$ is diagonal and has all non-zero entries equal to 1.

Question A6.

1. Find the Jordan normal form of the matrix $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \in M_3(\mathbb{C})$.

2. A square complex matrix A has characteristic polynomial $(X - i)^4(X - 4)^4$ and minimal polynomial $(X - i)^2(X - 4)^2$. The eigenspace corresponding to eigenvalue 4 has dimension 3. Write down all possible Jordan normal form matrices that are similar to A. (Up to re-arrangement of the blocks.)

Question A7.

1. Write the following element of S_5 as a product of disjoint cycles: $(1\,2\,3\,4)^{-1}(1\,5)$.

2. Calculate the order of $(12)(123)(54)(12) \in S_5$.

Question A8.

Let G be a group and $H \leqslant G$ a subgroup of G.

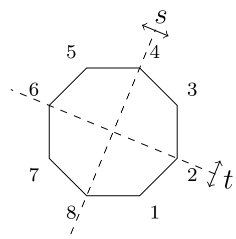
1. What does it mean to say that H is a *normal* subgroup of G?

Suppose now that $H = \langle \{aba^{-1}b^{-1} \mid a, b \in G\} \rangle \leqslant G$.

2. Show that H is normal.

Question A9.

Consider the subgroup H of D_8 given by $H = \langle s, t \rangle$ where $s, t \in D_8$ are the reflections shown in the figure on the right. The group H acts on the set of vertices $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$ of the octagon.



1. List the orbits of the action of H on X.

2. State the Orbit-Stabiliser relation and verify that it holds for each orbit of the action of H on X.

Question A10.

1. State Burnside's orbit counting lemma.

2. The sides of a square are coloured using three colours. Two such colourings are considered equivalent if one can be obtained from the other by rotating the square. How many different (i.e., non-equivalent) colourings are there?

Question B1.

1. Let $a, b \in \mathbb{Z}$ be relatively prime. Show that

   a) $\forall c \in \mathbb{Z}, \quad a \mid bc \implies a \mid c$

   b) $\forall c \in \mathbb{Z}, \quad (a \mid c \wedge b \mid c) \implies ab \mid c$

2. Let K be a field and $f(X) \in K[X]$. Show that $\forall k \in K, \quad f(k) = 0 \implies (X - k) \mid f(X)$

Question B2.

Let V be a finite dimensional vector space and $f : V \to V$ a linear transformation. Suppose that $v \in V$ and $n \in \mathbb{N}$ are such that $f^n(v) = 0$ and $f^{n-1}(v) \neq 0$.

1. Show that the set $S = \{v, f(v), \ldots, f^{n-1}(v)\}$ is linearly independent.

Let $W = \operatorname{Span}(S) \leqslant V$.

2. Show that W is f-invariant.

Let $g : W \to W$ be the restriction of f to W.

3. Show that g is nilpotent.
4. Show that there is a basis $\mathcal{B}$ of V and matrices A and B such that $[f]_{\mathcal{B}} = \begin{bmatrix} J(0,n) & A \\ 0 & B \end{bmatrix}$.

Question B3.

Let V be an inner product space and $f : V \to V$ a self-adjoint linear transformation.

1. Show that all eigenvalues of f are real.
2. Suppose that for all $v \in V$, $\langle f(v), v \rangle = 0$. Show that $f = 0$.
3. Suppose instead that V is finite dimensional and that $\langle f(v), v \rangle \geqslant 0$ for all $v \in V$. Show that $f = g^2$ for some self-adjoint linear transformation $g : V \to V$.

Question B4.

Let G be a group.

1. Show that if $\forall g \in G$, $g^2 = e$, then G is abelian.

Suppose now that G has order 8.

2. Show that if G is non-abelian, then G has an element of order 4.
3. Show that if G is abelian, then G is isomorphic to one of

$$\mathbb{Z}/8\mathbb{Z} \quad \text{or} \quad \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \quad \text{or} \quad \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

Question B5.

Give two non-isomorphic groups of size 10.
Show that there are only these two possibilities.

Question B6.

1. Show that there is only one group (up to isomorphism) of size 15.
2. Let G be a group of size 9.
   a) Show that the centre $Z(G)$ has size $|Z(G)| > 1$.
   b) Show that G is isomorphic to one of $\mathbb{Z}/9\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.
3. Let H be a group of size 42. Show that H has a normal subgroup $K \lhd H$ with $K \neq \{e\}$ and $K \neq H$.