MAST20022 Group Theory and Linear Algebra

Sample exam 4

Question A1.

1. Find the multiplicative inverse of $[9]_{14}$ in $\mathbb{Z}/14\mathbb{Z}$.
2. What can be said about the multiplicative inverse of $[6]_{14}$ in $\mathbb{Z}/14\mathbb{Z}$?
3. Using the Euclidean Algorithm, find $\gcd(299, 377)$.

Question A2.

1. Show that $\mathbb{Q}(i) = \{a + bi \mid a, b, \in \mathbb{Q}\} \subset \mathbb{C}$ is a field (using the usual operations on $\mathbb{C}$). (Hint: You may use that $\mathbb{C}$ is a field.)
2. a) Give the definition of what it means to say that a field is *algebraically closed*.
   b) Show that the field $\mathbb{Q}(i)$ is *not* algebraically closed.

Question A3.

Consider the matrix $M = \begin{bmatrix} i & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & i \end{bmatrix} \in M_3(\mathbb{C})$.

1. Find the minimal polynomial of M.
2. Use your answer for part (a) to determine whether M is diagonalizable. (Be sure to give a justification.)

Question A4.
Find all possible Jordan normal forms (up to permutation of Jordan blocks) for a matrix whose characteristic polynomial is $(X + 2)^2(X - 5)^3$

Question A5.
Let V be an inner product space and $f : V \to V$ a linear transformation.

1. Give the definition of the adjoint f^*.
2. Show that if $f = f^*$, then all eigenvalues of f are real.

Question A6.

Let V be a vector space and $U, W \leqslant V$ two subspaces of V.

1. Give the definition of what it means to say that $V = U \oplus W$.
2. Suppose that $V = U \oplus W$. Show that for all $v \in V$ there exist unique vectors $u \in U$ and $w \in W$ such that $v = u + w$.

Question A7.

1. Write the following element of S_5 as a product of disjoint cycles: $(1\,2\,3\,4)^{-1}(1\,5)$
2. Calculate the order of $(12)(123)(54)(12) \in S_5$.

Question A8.

Let G be a group and $H, K \leqslant G$ two subgroups.

1. Prove that $H \cap K$ is a subgroup of G.
2. Give and example to show that $H \cup K$ need not be a subgroup of G.

Question A9.

For each of the following pairs decide whether or not the two groups are isomorphic. You should justify your answers.

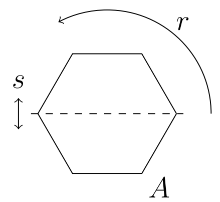
1. $(\mathbb{F}_5^\times, \times)$ and $(\mathbb{Z}/5\mathbb{Z}, +)$
2. $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$
3. $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ and D_5
4. $\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$ and $\mathbb{Z} \times (\mathbb{Z}/4\mathbb{Z})$

Question A10.

1. Let G be a finite group and $\varphi : G \to H$ a homomorphism.
   Prove that the order of $\varphi(G)$ divides the order of G.
2. How many homomorphisms are there from $\mathbb{Z}/3\mathbb{Z}$ to $S_3 \times \mathbb{Z}$?
   Be careful to justify your answer.

Question A11.

Let $s, r \in D_6$ be the symmetries of a regular hexagon corresponding to reflection across the line shown and rotation by $2\pi/3$ respectively. Consider the subgroup $H \leqslant D_6$ generated by $\{s, r\}$.



1. Find the orbit and stabilizer of the vertex A under the action of H.
   (Label the vertices of the hexagon A, B, C, D, E, F clockwise from the vertex A shown.)
2. Is the action of H transitive?

Question B1.

1. State the Orbit-Stabilizer relation.
2. Let G be a group of size 16 and X a set having 25 elements. Show that every action of G on X has a fixed point.
3. Suppose that a finite group G acts non-trivially on a finite set X. Let $n = |G|$ and $r = |X|$. Prove that if $n > r!$ then G has a normal subgroup $N \lhd G$ satisfying both $N \neq \{e\}$ and $N \neq G$.

Question B2.

Let V be a K-vector space and let $f : V \to V$ be a linear transformation.

1. Give the definition of an f-*invariant subspace* of V.
2. Let $p(X) \in K[X]$. Prove that $W = \ker(p(f))$ is an f-invariant subspace of V.
3. Suppose now that $p(f) = 0$ and that $p(X) = q_1(X)q_2(X)$ for some relatively prime $q_1(X), q_2(X) \in K[X]$. Show that $V = (\ker(q_1(f))) \oplus (\ker(q_2(f)))$.

Question B3.

1. Let V be an inner product space. Show that if $f : V \to V$ is a normal linear transformation, then $f(v) = 0$ if and only if $f^*(v) = 0$.
2. State the Spectral Theorem for linear transformations on a finite dimensional inner product space.
3. Let $A = \begin{bmatrix} 2 & i \\ i & 2 \end{bmatrix}$.
   a) Show that A is normal.
   b) Find a unitary matrix U such that U^*AU is diagonal.
   c) Show that there exists a matrix B such that $B^2 = A$.

Question B4.

The set

$$Q = \{1, -1, i, -i, j, -j, k, -k\}$$

has the structure of a group in which 1 is the identity element and the multiplication satisfies:

$$\begin{gathered} i^2 = j^2 = k^2 = -1, \qquad (-1)^2 = 1 \\ ij = k, \qquad jk = i, \qquad ki = j \\ -i = (-1)i, \qquad -j = (-1)j, \qquad -k = (-1)k \end{gathered}$$

(You do not have to prove that this is a group.)

1. Show that $ji = -k$ in Q.
2. Find the order of each element in Q.
3. Is Q isomorphic to D_4? Justify your answer.
4. Calculate the centre $Z(Q)$ of Q.
5. Determine which of the following groups is isomorphic to the quotient $Q/Z(Q)$:
   a) $\mathbb{Z}/2\mathbb{Z}$
   b) $\mathbb{Z}/4\mathbb{Z}$
   c) $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
   d) $\mathbb{Z}/8\mathbb{Z}$