MAST20022 Group Theory and Linear Algebra

Sample exam 3

Question 1. Show that the group $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and the group D_4 are not isomorphic.

Question 2. Describe all group homomorphisms $f \colon \mathbb{Z} \to \mathbf{Z}$.

Question 3. Let G be a group and let $g, x, y \in G$. Show that if $gx = gy$ then $x = y$.

Question 4. Let $: V \to V$ be a linear transformation on a finite dimensional inner product space V . Show that the adjoint f^* exists and is unique.

Question 5. Let $a, b, c \in \mathbb{C}$. Find the possible Jordan normal forms (up to reordering the Jordan blocks) of matrices that have characteristic polynomial $(x-a)(x-b)(x-c)$.

Question 6. Let $\mathbb{F}$ be a field and let $d, a \in \mathbb{F}[t]$. Define the ideal generated by d and "d divides a" and give some illustrative examples.

Question 7. Let $f \colon G \to H$ be a group homomorphism. Show that f is injective if and only if $\ker f = \{1\}$.

Question 8. Let $\mathbb{F}$ be a field. Define $\mathbb{F}[t]$ and $\mathbb{F}(t)$ and give some illustrative examples.

Question 9. Find the multiplicative inverse of 71 in $\mathbb{Z}/131\mathbb{Z}$.

Question 10. Define $\mathbb{R}^2$ and $\mathbb{E}^2$ and give some illustrative examples.

Question 11. Let G be the group of symmetries of the rectangle X with vertices $(2, 1)$, $(2, -1)$, $(-2, 1)$, $(-2, -1)$.

(a) Give geometric descriptions of the symmetries in G.

(b) Find the orbit and stabilizer of the point $Q = (2, 0)$ under the action of G on X.

(c) Check that your answers to parts (a) and (b) are consistent with the orbit-stabiliser theorem.

Question 12. Let V be the subspace of $\mathbb{R}^3$ spanned by the vectors $(1, 1, 0)$, $(0, 1, 2)$. Find the orthogonal complement of V, using the dot product as inner product on $\mathbb{R}^3$.

Question 13. Let $\mathcal{I}$ be the group of isometries of $\mathbb{E}^2$. Let P be a point of $\mathbb{E}^2$. Show that every element of $\mathcal{I}$ can be uniquely expressed as an isometry fixing P followed by a translation.

Question 14. Define the dihedral group D_n and give some illustrative examples.

Question 15. Let A be an $n \times n$ complex Hermitian matrix. Define a product on $\mathbb{C}^n$ by $(X, Y) = XAY^*$, where $X, Y \in \mathbb{C}^n$ are written as row vectors. Show that this is an inner product if all the eigenvalues of A are positive real numbers.

Question 16. Show that if $A = B^*B$, where B is any invertible $n \times n$ complex matrix, then A is a Hermitian matrix and all the eigenvalues of A are real and positive.