MAST20022 Group Theory and Linear Algebra

Sample exam 1

Question A1.

(a) Let a, b and c be integers. If $a|b$ and $a|c$, prove that $a^2|(b^2 + 3c^2)$.

(b) i. Use Euclid?s algorithm to find $d = \gcd(323, 377)$.

ii. Find integers x, y such that $323x + 377y = d$.

Question A2. Consider the set $\mathbb{Q}[i] = \{a + bi \mid |a, b \in \mathbb{Q}\}$, where $i^2 = 1$. Show that $\mathbb{Q}[i]$ forms a field under the usual operations of addition and multiplication of complex numbers.

Question A3. Let $f: V \to V$ be a linear transformation on an n-dimensional vector space with minimal polynomial $m(X) = X^n$.

(a) Show that there is a vector $v \in V$ such that $f^{n-1}(v) \neq 0$.

(b) Show that $\mathcal{B} = (f^{n-1}(v), f^{n-2}(v), \ldots, f^2(v), f(v), v)$ is a basis for V.

(c) Find the matrix of f with respect to the basis $\mathcal{B}$.

Question A4. Find the minimal polynomials and Jordan normal forms of the matrices:

$$B = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

Question A5. Which of the following pairs of matrices (over $\mathbb{C}$) are similar?

(a) $\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 5 \\ 0 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

Question A6.

(a) Find the order of each element in $\mathbb{Z}/10\mathbb{Z}$.

(b) Hence find all subgroups of $\mathbb{Z}/10\mathbb{Z}$.

Question A7. For each pair of groups, determine whether they are isomorphic or not and briefly justify your answer.

(a) $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and D_4

(b) $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ and $\mathbb{Z}/12\mathbb{Z}$

(c) $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$.

Question A8. Consider the subgroup $H = \langle (0, 2) \rangle$ of $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

(a) Write down the left cosets of H in G.

(b) Find the order of each element in the quotient group G/H.

(c) Identify the quotient group G/H. (Is it isomorphic to $\mathbb{Z}/4\mathbb{Z}$ or to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$?)

Question A9. Let V be an inner product space and let W be a subspace of V.

(a) Define the orthogonal complement $W^\perp$ of W.

(b) Show that $W \subseteq (W^\perp)^\perp$.

(c) Suppose now that V is finite-dimensional. Show that $W = (W^\perp)^\perp$.

Question A10. Let $GL_2(\mathbb{R})$ act on $\mathbb{R}^2$ in the usual way: $A \cdot v = Av$ for $A \in GL_2(\mathbb{R})$ and $v \in \mathbb{R}^2$. Describe the stabiliser and orbit of:

$$\text{(a)} \quad 0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{(b)} \quad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Question B1.

(a) State the Jordan normal form theorem.

(b) Give an explicit matrix over $\mathbb{F}_5$ that has a Jordan normal form. Justify your answer.

(c) Give an explicit matrix over $\mathbb{F}_5$ that does not have a Jordan normal form. Justify your answer.

(d) Let $V = \mathbb{C}[x]_{\leqslant 2}$ and consider a non-diagonalisable linear transformation $f\colon V \to V$ satisfying the conditions

$$\begin{aligned} (f - 2\mathrm{id}_V)^3 &= 0, \\ f(x-1) &= 2x - 2, \\ f(x^2+1) &= 2x^2 + 2. \end{aligned}$$

Find the Jordan normal form of f. Justify your answer.

Question B2. Let p be a prime number. Recall the groups of 2×2 matrices

$$GL_2(\mathbb{F}_p) = \{A \in M_2(\mathbb{F}_p) \mid \det(A) \neq 0\}$$
$$SL_2(\mathbb{F}_p) = \{A \in M_2(\mathbb{F}_p) \mid \det(A) = 1\}.$$

(a) Prove that $\#GL_2(\mathbb{F}_p) = (p^2 - 1)(p^2 - p)$.

(b) Use the determinant group homomorphism to find the cardinality $\#SL_2(\mathbb{F}_p)$.

(c) Consider the subset

$$H = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \mid x \in \mathbb{F}_p \right\}.$$

Prove that H is a subgroup of $SL_2(\mathbb{F}_p)$.

(d) Is H a normal subgroup? Justify your answer.

(e) Prove that H is isomorphic to the group $(\mathbb{F}_p, +)$.

(f) Write down an explicit element of order p of $SL_2(\mathbb{F}_p)$. Justify your answer.

(g) How many p-Sylow subgroups does $SL_2(\mathbb{F}_p)$ have? Justify your answer.

Question B3.

(a) State the Spectral Theorem for complex matrices.

(b) Show that the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

is normal.

(c) Find a complex matrix square root of A, i.e. a complex matrix B such that $B^2 = A$.

Question B4.

(a) Consider the action of the group D_4 on $\mathbb{R}^2$ defined by

$$r \cdot v = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} v, s \cdot v = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} v,$$

i. What is the vector

$$(sr^3) \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix}?$$

ii. What cardinalities can the orbits of this action of D_4 have? Give an explicit example for each cardinality.

(b) i. State Burnside?s Lemma for the number of orbits of the action of a finite group on a finite set.

ii. Find the number of 3×3 squares containing only 0?s and 1?s, up to D_4 symmetry.