Problem sheet 3

Vector spaces, bases, linear transformations

Vocabulary

(1) Define a vector space and give some illustrative examples.

(2) Define subspace and the intersection and sum of subspaces and give some illustrative examples.

(3) Define a similar matrices and give some illustrative examples.

(4) Define the change of basis matrix and give some illustrative examples.

(5) Define the kernel, image, rank and nullity of a linear transformation and give some illustrative examples.

(6) Define a linear transformation and give some illustrative examples.

(7) Define basis and dimension and give some illustrative examples.

(8) Define linearly dependent and linearly independent vectors and give some illustrative examples.

(9) Define linear combination, linearly dependent and linearly independent and give some illustrative examples.

Results

(1) Show that any subset of a linearly independent set is also linearly independent.

(2) Let $\mathbb{F}$ be a field and let $E_{ij} \in M_{m\times n}(\mathbb{F})$ be the matrix with 1 in the (i, j)-position and 0 elsewhere. Show that $\{E_{ij} \mid i \in \{1, \dots, m\} \text{ and } j \in \{1, \dots, n\}\}$ is a basis of $M_{m\times n}(\mathbb{F})$.

(3) Let $m, n \in \mathbb{Z}_{>0}$. Define $M_{m\times n}(\mathbb{R})$, addition and scalar multiplication, and show that $M_{m\times n}(\mathbb{R})$ is a vector space.

(4) Let $\mathbb{F}$ be a field. Define $\mathbb{F}[t]$, addition and scalar multiplication, and show that $\mathbb{F}[t]$ is a vector space.

(5) Let S be a set and let $\mathbb{F}$ be a field. Define addition and scalar multiplication on $\mathcal{F}(S, \mathbb{F}) = \{f : S \to \mathbb{F}\}$, the set of functions from S to $\mathbb{F}$, and show that $\mathcal{F}(S, \mathbb{F})$ is a vector space.

(6) Let B_V and B'_V be bases of V and let P be the change of basis matrix from B_V to B'_V. Let B_W and B'_W be bases of W and let Q be the change of basis matrix from B_W to B'_W. Let $f : V \to W$ be a linear transformation and let A be the matrix of f with respect to the bases B_V and B_W.

(a) Show that P and Q are invertible.

(b) Show that the matrix of f with respect to the bases B_V and B_W is QAP^{-1}.

(7) Let $f: V \to W$ be a linear transformation.

(a) Show that the nullspace of f is a subspace of V.

(b) Show that the image of f is a subspace of W.

(8) Let $f: V \to W$ be a linear transformation and assume that V is finite dimensional. Show that the nullity of f plus the rank of f is equal to the dimension of V.

(9) Let U and W be subspaces of a vector space V and assume that $U+W$ is finite dimensional. Then

$$\dim(U+W)\dim(U \cap W) = \dim(U) + \dim(W).$$

(10) Show that every vector space has a basis. In fact, every spanning set contains a basis and every linearly independent set can be extended to a basis.

(11) Show that if B_1 and B_2 are two bases of a vector space then they have the same number of elements. (This means that you need to show that there exists a bijective function $f: B_1 \to B_2$.)

(12) Show that a subset S of a vector space V is linearly dependent if and only if there exists $s \in S$ which is a linear combination of the others.

(13) If S is a non-empty subset of V, then span(S) is a subspace of V,

(14) Let V be a vector space over $\mathbb{F}$. A subset W of V is a subspace if and only if the following three conditions are satisfied:

(1) W is non-empty,

(2) If $u, w \in W$ then $u + w \in W$.

(3) if $a \in \mathbb{F}$ and $w \in W$ then $aw \in W$.

(15) Let $f: V \to V$ be a linear transformation on a finite dimensional vector space V. Show that the nullity of f is zero if and only if f is surjective.

(16) Let V be a vector space. Show that if U and W are subspaces of V then $U + W = \{u + w \mid u \in U \text{ and } w \in W\}$ is a subspace of V.

(17) Let V be a vector space. Show that if U and W are subspaces of V then $U \cap W$ is a subspace of V.

(18) Let V be a vector space. Show that if U and W are subspaces of V and $U \cup W = V$ then $U = V$ or $W = V$.

Examples and computations

(1) Define $\mathbb{R}^3$, addition and scalar multiplication, and show that $\mathbb{R}^3$ is a vector space.

(2) Let $\mathbb{F}$ be a field and $n \in \mathbb{Z}_{.0}$. Define $\mathbb{F}^n$, addition and scalar multiplication, and show that $\mathbb{F}^n$ is a vector space.

(3) Let $\mathbb{F}$ be a field. Define $cP_n(\mathbb{F}) = \{a_0 + a_1 t + \cdots + a_n t^n \mid a_0, a_1, \dots, a_n \in \mathbb{F}\}$, addition and scalar multiplication, and show that $\mathcal{P}_n(\mathbb{F})$ is a vector space.

(4) Define addition and scalar multiplication on $\mathcal{F}(\mathbb{R}, \mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R}\}$, the set of functions from $\mathbb{R}$ to $\mathbb{R}$, and show that $\mathcal{F}(\mathbb{R}, \mathbb{R})$ is a vector space.

(5) Define addition and scalar multiplication on the set $\mathcal{S}$ of solutions y of the differential equation

$$\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 23y = 0$$

and show that $\mathcal{S}$ is a vector space.

(6) Define addition and scalar multiplication on the set

$$\ell^\infty = \{(a_1, a_2, \dots) \mid a_i \in \mathbb{R}\}$$

and show that ℓ^∞ is a vector space.

(7) Define addition and scalar multiplication on the set

$$c_0 = \{(a_1, a_2, \dots) \mid a_i \in \mathbb{R} \text{ and } \lim_{i\to\infty} a_i = 0\}$$

and show that c_0 is a vector space.

(8) Show that $\{(a_1, a_2, \dots) \mid a_i \in \mathbb{R} \text{ and } \lim_{i\to\infty} a_i = 1\}$ is not a subspace of ℓ^∞.

(9) Show that $W = \{(a, b, c) \mid a, b, c \in \mathbb{R} \text{ and } a + b + c = 0\}$ is a subspace of $\mathbb{R}^3$.

(10) Show that the set of matrices of trace zero is a subspace of the vector space $M_n(\mathbb{R})$.

(11) Show that the set of polynomials with zero constant term is a subspace of the vector space $\mathbb{R}[t]$.

(12) Show that the set of differentiable functions is a subspace of the vector space $\mathcal{F}(\mathbb{R}, \mathbb{R})$ of functions from $\mathbb{R}$ to $\mathbb{R}$.

(13) Show that the set c_0 of sequences such that $\lim_{i\to\infty} a_i = 0$ is a subspace of the vector space of sequences ℓ^∞.

(14) Show that the set of linear combinations of the vectors $(1, -2, 3)$ and $(0, 2, 1)$ in $\mathbb{R}^3$ is the set $\{(a, -2a + 2b, 3 + b \mid a, b \in \mathbb{R}\}$.

(15) Show that the set of linear combinations of the matrices

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

in $M_3(\mathbb{R})$ is the set of matrices of the form

$$\begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}, \qquad \text{where } a, b, c \in \mathbb{R}.$$

(16) Show that the set $\{(2,1,3),(2,-1,0),(-1,8,9)\}$ is linearly dependent in $\mathbb{R}^3$.

(17) Show that the set $\{1, x, x^2, 1+x^3\}$ is linearly independent in $\mathbb{R}[x]$.

(18) Show that the set $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & -29 \\ 0 & 0 \end{pmatrix}\right\}$ is linearly dependent in $M_2(\mathbb{R})$.

(19) Let $\mathbb{F}$ be a field and let $n \in \mathbb{Z}_{>0}$. Show that $\{e_1 = (1,0,0,\ldots,0), e_1 = (0,1,0,\ldots,0), \ldots, e_n = (0,0,0,\ldots,1)\}$ is a basis of $\mathbb{F}^n$.

(20) Show that the set $\{(2,1,3),(1,2,3),(1,0,0)\}$ is a basis of $\mathbb{R}^3$.

(21) Show that the set $\{1, x, x^2, 1+x^3\}$ is a basis of the vector space of polynomials with coefficients in $\mathbb{R}$ of degree $\leqslant 3$.

(22) Show that the set $\{1, x, x^2, x^3, \ldots\}$ is a basis of the vector space $\mathbb{R}[x]$.

(23) Show that $\mathbb{R}^3$ has dimension 3.

(24) Let $\mathbb{F}$ be a field and let $n \in \mathbb{Z}_{>0}$. Show that $\mathbb{F}^n$ has dimension n.

(25) Let $m, n \in \mathbb{Z}_{>0}$. Show that $M_{m\times n}(\mathbb{R})$ has dimension mn.

(26) Show that the set of polynomials with coefficients in $\mathbb{R}$ and degree $\leqslant n$ has dimension $n+1$.

(27) Show that the vector space $\mathcal{S}$ of solutions y

$$\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 23y = 0$$

has dimension 2.

(28) Show that ℓ^∞ has infinite dimension.

(29) Show that c_0 has infinite dimension.

(30) Let $\mathbb{F}$ be a field. Show that $\mathbb{F}[t]$ has infinite dimension.

(31) Show that rotation about the origin through a fixed angle θ is a linear transformation on $\mathbb{R}^2$.

(32) Show that rotation about any line by a fixed angle θ is a linear transformation on $\mathbb{R}^3$.

(33) Show that differentiation with respect to t is a linear transformation on $\mathbb{R}[t]$.

(34) Let $C(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ if continuous}\}$, a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$. Let $I : C(\mathbb{R}) \to C(\mathbb{R})$ be given by

$$I(f)(t) = \int_0^t f(x)dx.$$

Show that I is a linear transformation.

(35) Show that the functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ and $g(x) = x + 2$ are not linear transforrnations.

(36) Show that rotation in $\mathbb{R}^2$ has kernel $\{0\}$ and image $\mathbb{R}^2$.

(37) Show that differentiation with respect to x on $\mathbb{R}[x]$ has kernel $\mathbb{R} \cdot 1$ and image $\mathbb{R}[x]$.

(38) Rotation about the origin through a fixed angle θ is a linear transformation f on $\mathbb{R}^2$. Find the matrix of f with respect to the basis $\{(1, 0), (0, 1)\}$.

(39) Differentiation with respect to t is a linear transformation f on $\mathbb{R}[t]$. Find the matrix of f with respect to the basis $\{1, t, t^2, \ldots\}$.

(40) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $f(x, y) = (3x - y, -x + 3y)$. Let $\mathcal{B} = \{(1, 0), (0, 1)\}$ and let $\mathcal{C} = \{(1, 1), (-1, 1)\}$. Find the change of basis matrix P from $\mathcal{B}$ to $\mathcal{C}$ and the change of basis matrix Q from $\mathcal{C}$ to $\mathcal{B}$. Find the matrix A of f with respect to the basis $\mathcal{B}$ and the matrix B of f with respect to the basis $\mathcal{C}$. Verify that $A = PBQ$.

(41) In the vector space $(\mathbb{Z}/7\mathbb{Z})^4$ determine whether the set $\{(1, 3, 0, 2), (2, 1, 3, 0)\}$ is linearly dependent and whether it is a basis.

(42) In the vector space $(\mathbb{Z}/7\mathbb{Z})^4$ determine whether the set $\{(1, 2, 3, 1), (4, 6, 2, 0,), (0, 1, 5, 1)\}$ is linearly dependent and whether it is a basis.

(43) In the vector space $(\mathbb{Z}/7\mathbb{Z})^4$ determine whether the set

$$\{(1, 2, 3, 1), (4, 6, 2, 0,), (0, 1, 5, 2), (0, 1, 1, 0), (0, 1, 0, 1)\}$$

is linearly dependent and whether it is a basis.

(44) In the vector space $M_2(\mathbb{R})$ determine whether the set

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

is linearly dependent and whether it is a basis.

(45) In the vector space $M_2(\mathbb{R})$ determine whether the set

$$\left\{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right\}$$

is linearly dependent and whether it is a basis.

(46) In the vector space $M_2(\mathbb{R})$ determine whether the set

$$\left\{\begin{pmatrix}2 & 0\\ 1 & 4\end{pmatrix}, \begin{pmatrix}1 & 3\\ 0 & 2\end{pmatrix}, \begin{pmatrix}4 & -6\\ 3 & 8\end{pmatrix}\right\}$$

is linearly dependent and whether it is a basis.

(47) What is the dimension of the space $M_3(\mathbb{Z}/5\mathbb{Z})$?

(48) Let $B = \begin{pmatrix}2 & 1\\ 3 & -1\end{pmatrix}$. Show that the function $g\colon M_2(\mathbb{R}) \to M_2(\mathbb{R})$ given by $g(A) = AB$, , is a linear transformation.

(49) Let $B = \begin{pmatrix}2 & 1\\ 3 & -1\end{pmatrix}$. Find the matrix of the linear transformation $g\colon M_2(\mathbb{R}) \to M_2(\mathbb{R})$ given by $g(A) = AB$, with respect to the basis

$$\left\{E_{11} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, E_{12} = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, E_{21} = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, E_{22} = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix},\right\}.$$

(50) Find the matrix, with respect to the standard basis of $\mathbb{R}^2$, of the reflection in the x-axis. Let $a, b, c, d \in \mathbb{R}$ such that $ad - bc \neq 0$. Let B be the basis of $\mathbb{R}^2$ given by $\{(a, b), (c, d)\}$. Determine the change of basis matrix from the standard basis of $\mathbb{R}^2$ to B and use it to calculate the matrix of the reflection with respect to the basis B.

(51) Calculate the nullity and rank of the linear transformation f on $\mathbb{R}^3$ given by

$$f(e_1) = e_1 - e_2, \qquad f(e_2) = e_2 - e_3, \qquad f(e_3) = e_1 - e_3,$$

where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$.

(52) Calculate the nullity and rank of the linear transformation f on $(\mathbb{Z}/7\mathbb{Z})^2$ given by

$$f(1, 0, 0) = (1, 2, 3), \qquad f(0, 1, 0) = (3, 4, 5), \qquad f(0, 0, 1) = (5, 1, 4).$$

(53) Determine whether the set of upper triangular matrices with real entries

$$\{A = (a_{ij} \in M_3(\mathbb{R}) \mid a_{ij} = 0 \text{ for } i > j\}$$

is a vector space over $\mathbb{R}$.

(54) Determine whether the set of functions $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(0) \geqslant 0$ is a vector space over $\mathbb{R}$.

(55) Consider the subset $S = \{(1, 3), (3, 4), (2, 3)\}$ in $(\mathbb{Z}/5\mathbb{Z})^2$.

(i) Does S span $(\mathbb{Z}/5\mathbb{Z})^2$?

(ii) Is S linearly independent?

a)](iii)] Find a subset of S which is a basis of $(\mathbb{Z}/5\mathbb{Z})^2$.

(56) Let U, W be 3-dimensional subspaces of $\mathbb{R}^5$. Show that $U \cap W$ contains a non-zero vector.

(57) Define $f\colon M_3(\mathbb{R}) \to M_3(\mathbb{R})$ by $f(A) = A + A^t$, where A^t is the transpose of A.

(i) Show that f is a linear transformation.

(ii) Describe the kernel and image of f

(iii) Find bases for these spaces, and verify that the rank-nullity formula holds.

(58) Are the following sets of functions from $\mathbb{R}$ to $\mathbb{R}$ linearly independent?

(i) $\{1, \sin^2 x, \cos^2 x\}$,

(ii) $\{1, \sin(2x), \cos(2x)\}$.

(59) Show that $\{1, 2, 3\}$ is linearly independent over the field $\mathbb{Q}$.

(60) Let $\beta = \sqrt[3]{2}$. Then $V = \{x + y\beta + z\beta^2 \mid x, y, z \in \mathbb{Q}\}$ is a vector space over the field $\mathbb{Q}$.

(a) Show that V is closed under multiplication.

(b) Let α be a nonzero element of V, and let $f\colon V \to V$ be multiplication by α, $f(v) = \alpha v$. Show that f is a linear transformation and determine the kernel and the image of f.

(c) Show that V is a field.