The *quaternion group* is

$Q = \{1, -1, i, -i, j, -j, k, -k\}$ with

$Q \times Q \to Q$ given by

	1	-1	i	$-i$	j	$-j$	k	$-k$
1	1	-1	i	$-i$	j	$-j$	k	$-k$
-1	-1	1	$-i$	i	$-j$	j	$-k$	k
i	i	$-i$	-1	1	k	$-k$	$-j$	j
$-i$	$-i$	i	1	-1	$-k$	k	j	$-j$
j	j	$-j$	$-k$	k	-1	1	i	$-i$
$-j$	$-j$	j	k	$-k$	1	-1	$-i$	i
k	k	$-k$	j	$-j$	$-i$	i	-1	1
$-k$	$-k$	k	$-j$	j	i	$-i$	1	-1

so that

$i^2 = -1$, $j^2 = -1$, $k^2 = -1$, $ij = k$, $jk = i$, $ki = j$.

HW: Calculate the orders of the elements of Q, and the subgroups and the conjugacy classes.

HW: Prove that Q is not isomorphic to D_4.

The group $\mu_4 = \{1, -1, i, -i\}$ of 4^{th} roots of unity is

$\mu_4 = \{1, i, i^2, i^3\}$ with $i^4 = 1$.

The <u>complex numbers</u> is

$$\mathbb{C} = \mathbb{R}\text{span}\{1, -1, i, -i\} = \{x+iy \mid x, y \in \mathbb{R}\}$$ with multiplication

determined by the multiplication on μ_4 and the distributive law.

The <u>Hamiltonians</u>, or <u>quaternions</u>, is

$$\mathbb{H} = \mathbb{R}\text{-span}\{1, -1, i, -i, j, -j, k, -k\} = \{t + xi + yj + zk \mid t, x, y, z \in \mathbb{R}\}$$ with multiplication

determined by the multiplication on Q. and the distributive law.

<u>HW</u>: (a) Show that $\mathbb{C}$ is a commutative ring.
(b) Show that $\mathbb{H}$ is a noncommutative ring

Let $\mathfrak{u}_1(\mathbb{H}) = \{xi + yj + zk \mid x, y, z \in \mathbb{R}\}$ and define

$$\begin{aligned} \mathfrak{u}_1(\mathbb{H}) \times \mathfrak{u}_1(\mathbb{H}) &\to \mathbb{R} \\ (v_1, v_2) &\mapsto v_1 \cdot v_2 \end{aligned} \quad \text{and} \quad \begin{aligned} \mathfrak{u}_1(\mathbb{H}) \times \mathfrak{u}_1(\mathbb{H}) &\to \mathfrak{u}_1(\mathbb{H}) \\ (v_1, v_2) &\mapsto v_1 \times v_2 \end{aligned}$$

by

$$(x_1 i + y_1 j + z_1 k) \cdot (x_2 i + y_2 j + z_2 k) = x_1 x_2 + y_1 y_2 + z_1 z_2$$

and

$$(x_1 i + y_1 j + z_1 k) \times (x_2 i + y_2 j + z_2 k) = \begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

$$= (y_1 z_2 - y_2 z_1) i - (x_1 z_2 - x_2 z_1) j + (x_1 y_2 - x_2 y_1) k.$$

<u>Proposition</u> The multiplication on $\mathbb{H}$ is given by

$$(t_1 + v_1)(t_2 + v_2) = (t_1 t_2 - v_1 \cdot v_2) + (t_1 v_2 + t_2 v_1 + v_1 \times v_2).$$

for $t_1, t_2 \in \mathbb{R}$ and $v_1, v_2 \in \mathcal{U}_1(\mathbb{H})$

Define $\bar{\ } : \mathbb{H} \to \mathbb{H}$ by

$$\overline{t + xi + yj + zk} = t - xi - yj - zk$$

for $t, x, y, z \in \mathbb{R}$. Define $\| \ \| : \mathbb{H} \to \mathbb{R}_{\geq 0}$ by

$$\| t + xi + yj + zk \| = \sqrt{t^2 + x^2 + y^2 + z^2}.$$

<u>Theorem</u> Let $h \in \mathbb{H}$.

(a) $h \bar{h} = \|h\|^2$

(b) If $h \in \mathbb{H}$ and $h \neq 0$ then there exists $h^{-1} \in \mathbb{H}$ with $h h^{-1} = 1 = h^{-1} h$.

So $\mathbb{H}$ is a "noncommutative field".

Polar form for elements of $\mathbb{H}$

Let $w = ai + bj + ck$ with

$$a, b, c \in \mathbb{R} \text{ and } a^2 + b^2 + c^2 = 1.$$

Then

$$w^2 = (0 - w \cdot w) + 0w + 0w + w \times w = -1.$$

and

$$e^{w\theta} = \cos\theta + w\sin\theta$$

$$= \cos\theta + a\sin\theta\, i + b\sin\theta\, j + c\sin\theta\, k.$$

and

$$re^{w\theta} = r\cos\theta + ar\sin\theta\, i + br\sin\theta\, j + cr\sin\theta\, k$$

$$= t + xi + yj + zk$$

provided

$$r = \sqrt{x^2 + y^2 + z^2 + t^2}$$

$$\cos\theta = \frac{t}{r}$$

$$\sin^2\theta = \frac{x^2 + y^2 + z^2}{r^2}$$

$$a = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$b = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

$$c = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$t = r\cos\theta$$

$$x = ar\sin\theta$$

$$y = br\sin\theta$$

$$z = cr\sin\theta$$