Let G be a group and let H be a subgroup.
The <u>set of cosets of H in G</u> is

$$G/H = \{gH \mid g \in G\} \quad \text{where}$$

$$gH = \{gh \mid h \in H\}.$$

<u>Theorem</u> $\mathrm{Card}(G) = \mathrm{Card}(G/H)\,\mathrm{Card}(H)$ and, even better,

the cosets partition G and

$\mathrm{Card}(gH) = \mathrm{Card}(H)$.

Can we multiply cosets?

<u>Proposition</u> Let G be a group and let H be a subgroup. Then

$$\begin{array}{ccc} G/H \times G/H & \to & G/H \\ (aH, bH) & \mapsto & abH \end{array} \quad \text{is a function.}$$

if and only if H is normal.

A subgroup H of G is <u>normal</u> if H satisfies:
if $h \in H$ and $g \in G$ then $g^{-1}hg \in H$.

Example $G = S_3 = D_3 = \{1, r, r^2, s, rs, r^2s\}$

with $r^3 = 1$, $s^2 = 1$ and $sr = r^{-1}s$.

Let $H = \{1, s\}$.

Then

$$H = \{1, s\} \qquad rH = \{r, rs\} \qquad r^2H = \{r^2, r^2s\}$$
$$= sH, \qquad = rsH, \qquad = r^2sH$$

and $G/H = \{H, rH, r^2H\}$.

Let

$$\mu : G/H \times G/H \longrightarrow G/H$$
$$(aH, bH) \longrightarrow abH.$$

Then

$$\mu(H, rH) = 1 \cdot rH = rH = \{r, rs\}$$
$$\stackrel{?}{=} \mu(sH, rH) = srH = r^2sH = \{r^2, r^2s\}$$

OOPS. So μ is not a function.

in the same way that

$$\mathbb{R}_{>0} \longrightarrow \mathbb{R}$$
$$x \longmapsto \sqrt{x}$$ is not a function

since $\sqrt{9} = 3 \neq \sqrt{9} = -3$.

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Proof $\Rightarrow$ To show: If $\mu: G/H \times G/H \to G/H$ is
$(aH, bH) \mapsto abH$
a function then H is normal.

Assume μ is a function

To show: If $h \in H$ and $g \in G$ then $g^{-1}hg \in H$.

Assume $h \in H$ and $g \in G$.

To show: $g^{-1}hg \in H$.

$$
\begin{aligned}
g^{-1}hg \in g^{-1}hgH &= \mu(g^{-1}hH, gH) \\
&= \mu(g^{-1}H, gH) = g^{-1}gH \\
&= H.
\end{aligned}
$$

So H is normal.

$\Leftarrow$ To show: If H is normal then μ is a function.

Assume H is normal.

To show: If $a_1, a_2, b_1, b_2 \in G$ and $a_1H = a_2H$ and $b_1H = b_2H$ then $a_1b_1H = a_2b_2H$.

Assume $a_1, a_2, b_1, b_2 \in G$ and $a_1H = a_2H$ and $b_1H = b_2H$.

Since $a_1 \in a_2H$ there exists $h_1 \in H$ such that $a_1 = a_2h_1$

Since $b_1 \in b_2H$ there exists $h_2 \in H$ such that $b_1 = b_2h_2$.

Then
$$a_1 b_1 = a_2 h_1 b_2 h_2 = a_2 b_2 b_2^{-1} h_1 b_2 h_2$$
$$= a_2 b_2 (b_2^{-1} h_1 b_2) h_2.$$

Since H is normal $b_2^{-1} h_1 b_2 \in H$.

So $b_2^{-1} h_1 b_2 h_2 \in H$, since H is a subgroup.

So $a_1 b_1 = a_2 b_2 (b_2^{-1} h_1 b_2) h_2 \in a_2 b_2 H$.

So $a_1 b_1 H \cap a_2 b_2 H \neq \emptyset$.

So $a_1 b_1 H = a_2 b_2 H$. //

<u>Proposition</u> Let G be a group and let N be a normal subgroup of G. Then

$$G/N \text{ with } \begin{matrix} G/N \times G/N & \to & G/N \\ (aN, bN) & \mapsto & abN \end{matrix}$$

is a group.

<u>Proof</u> Since N is normal the operation is a function.

To show: (a) If $a_1, a_2, a_3 \in G$ then
$$(a_1 N \cdot a_2 N) a_3 N = a_1 N \cdot (a_2 N \cdot a_3 N).$$

(b) There exists $e \in G$ such that if $g \in G$ then $eN \cdot gN = gN$ and
$$gN \cdot eN = gN.$$

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(c) If $g \in G$ then there exists $b \in G$ such that $gN \cdot bN = eN$ and $bN \cdot gN = eN$

(a) Assume $a_1, a_2, a_3 \in G$. Then

$$(a_1N \cdot a_2N) \cdot a_3N = a_1a_2N \cdot a_3N = (a_1a_2)a_3N,$$

$$a_1N(a_2N \cdot a_3N) = a_1N \cdot a_2a_3N = a_1(a_2a_3)N$$

and $(a_1a_2)a_3 = a_1(a_2a_3)$ by associativity in G.

(b) Let $e = 1$.

Assume $g \in G$. Then

$$eN \cdot gN = egN = 1 \cdot gN,$$

$$gN \cdot eN = geN = g \cdot 1N,$$

and $1 \cdot g = g$ and $g \cdot 1 = g$ since 1 is the identity in G.

(c) Assume $g \in G$.

Let $b = g^{-1}$. Then

$$gN \cdot bN = gbN = gg^{-1}N = 1N = eN \text{ and}$$

$$bN \cdot gN = bgN = g^{-1}gN = 1N = eN.$$

So G/N with $\begin{array}{ccc} G/N \times G/N & \to & G/N \\ (aN, bN) & \mapsto & abN \end{array}$ is a group.