Let G be a group. Let S be a set.
~~The~~ An action of G on S is a function

$$\begin{aligned} G\times S &\longrightarrow S \\ (g,x) &\longmapsto g\cdot x \end{aligned}$$ such that

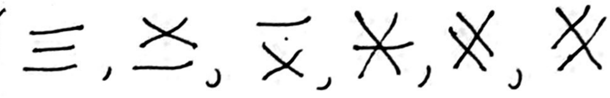
(a) If $g_1, g_2 \in G$ and $x \in S$ then
$$g_1\cdot(g_2\cdot x) = (g_1g_2)\cdot x$$
(b) If $x\in S$ then $1\cdot x = x$

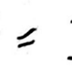
S (with its action of G) is a G-set (analogous to an $\mathbb{F}$-vector space).

Let S be a G-set and let $x \in S$.
The stabilizer of x is
$$\mathrm{Stab}_G(x) = \{g\in G \mid g\cdot x = x\}$$
The orbit of x is
$$G\cdot x = \{g\cdot x \mid g\in G\}.$$

Let $f: G\to H$ be a homomorphism.
$$\ker f = \{g\in G \mid f(g) = 1\}$$
$$\operatorname{im} f = \{f(g) \mid g\in G\}.$$

A choice of generators and relations for S_3

$S_3 = \{$  $\}$

Let $1 =$ , $r =$ , $s =$ 

Then $r^2 =$  $=$  $r^3 = r^2 r =$  $=$  $= 1$

$sr =$  $=$  $s^2 =$  $=$  $= 1$

$sr^2 =$  $=$  $r^2 s =$  $=$ 

So $S_3 = \{1, r, r^2, s, sr, sr^2\}$ with

$r^3 = 1, \quad s^2 = 1, \quad sr = r^{-1}s.$

(Note $r^{-1} = r^2$.)

Proposition Let S be a G-set. Let $x \in S$ and $g \in G$.

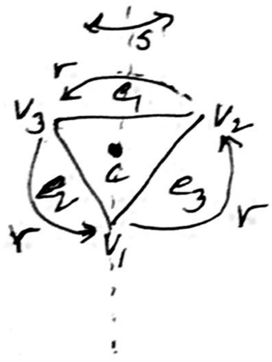
(a) $\mathrm{Stab}_G(x)$ is a subgroup of G

(b) $G \cdot x$ is a subset of S

(c) $\mathrm{Stab}_G(g \cdot x) = g\, \mathrm{Stab}_G(x)\, g^{-1}$

(d) The orbits partition S.

<u>Example</u> $S_3 = D_3$ acting on a triangle



$S = \{c, v_1, v_2, v_3, e_1, e_2, e_3\}$

s is reflection in the vertical

r is rotation by $\frac{2\pi}{3}$

Then

<u>Orbits</u> Let $G = S_3 = D_3$

$G \cdot c = \{c\}$

$G \cdot v_1 = G \cdot v_2 = G \cdot v_3$
$= \{v_1, v_2, v_3\}$

$G \cdot e_1 = G \cdot e_2 = G \cdot e_3$
$= \{e_1, e_2, e_3\}.$

<u>Stabilizers</u>

$Stab_G(c) = S_3 = \{1, s, sr, sr^2, r, r^2\}$

$Stab_G(v_1) = \{1, s\}$

$Stab_G(v_2) = \{1, sr^2\}$

$Stab_G(v_3) = \{1, sr\}$

$Stab_G(e_1) = \{1, s\}$

$Stab_G(e_2) = \{1, sr^2\}$

$Stab_G(e_3) = \{1, sr\}.$

(c) To show: $\mathrm{Stab}_G(g\cdot x) = g\,\mathrm{Stab}_G(x)g^{-1}$.

To show: (ca) $\mathrm{Stab}_G(g\cdot x) \subseteq g\,\mathrm{Stab}_G(x)g^{-1}$

(cb) $g\,\mathrm{Stab}_G(x)g^{-1} \subseteq \mathrm{Stab}_G(g\cdot x)$.

(ca) Assume $z \in \mathrm{Stab}_G(g\cdot x)$.

To show: $z \in g\,\mathrm{Stab}_G(x)g^{-1}$.

To show: $g^{-1}zg \in \mathrm{Stab}_G(x)$.

Since

$$g^{-1}zg\cdot x = g^{-1}z\cdot(g\cdot x) = g^{-1}\cdot(g\cdot x) = (g^{-1}g)\cdot x$$
$$= 1\cdot x = x$$

then $g^{-1}zg \in \mathrm{Stab}_G(x)$.

(cb) Assume $y \in g\,\mathrm{Stab}_G(x)g^{-1}$.

To show: $y \in \mathrm{Stab}_G(g\cdot x)$.

Since $y \in g\,\mathrm{Stab}_G(x)g^{-1}$ then there exists $k \in \mathrm{Stab}_G(x)$ such that $y = gkg^{-1}$.

Since

$$gkg^{-1}\cdot(g\cdot x) = gkg^{-1}g\cdot x = gk\cdot x$$
$$= g\cdot x$$

then $gkg^{-1} \in \mathrm{Stab}_G(g\cdot x)$.

So $\mathrm{Stab}_G(g\cdot x) = g\,\mathrm{Stab}_G(x)g^{-1}$.

(d) To show: (da) If $x, y \in S$ and $G\cdot x \cap G\cdot y \neq \emptyset$ then $G\cdot x = G\cdot y$.

(db) $S = \bigcup_{x \in S} G\cdot x$.

(da) Assume $x, y \in S$ and $G\cdot x \cap G\cdot y \neq \emptyset$.

Since $G\cdot x \cap G\cdot y \neq \emptyset$ then there exists $z \in S$ such that $z \in G\cdot x$ and $z \in G\cdot y$.

So there exists $g_1, g_2 \in G$ such that $z = g_1\cdot x$ and $z = g_2\cdot y$.

So $x = g_1^{-1}z = g_1^{-1}g_2\cdot y$ and $y = g_2^{-1}z = g_2^{-1}g_1\cdot x$

To show: $G\cdot x = G\cdot y$.

To show: (daa) $G\cdot x \subseteq G\cdot y$

(dab) $G\cdot y \subseteq G\cdot x$.

(daa)
Let $a \in G\cdot x$.

To show: $a \in G\cdot y$.

There exists $k \in G$ such that $a = k\cdot x$.

So $a = k\cdot x = k g_1^{-1} g_2\cdot y \in G\cdot y$.

So $G\cdot x \subseteq G\cdot y$.

(dab) To show: $G\cdot y \subseteq G\cdot x$

Let $b \in G\cdot y$. To show: $b \in G\cdot x$.

There exists $g \in G$ such that $b = g\cdot y$.

So $b = g\cdot y = g g_2^{-1} g_1 \cdot x \in G\cdot x$.

So $G\cdot y \subseteq G\cdot x$.

So $G\cdot x = G\cdot y$.

(db) To show: $S = \bigcup_{x \in S} G\cdot x$.

To show: (dba) $\bigcup_{x\in S} G\cdot x \subseteq S$

(dbb) $S \subseteq \bigcup_{x \in S} G\cdot x$.

(dba) Let $x \in S$. Since

$G\cdot x = \{g\cdot x \mid g \in G\} \subseteq S$ then

$\bigcup_{x\in S} G\cdot x \subseteq S$.

(dbb) Let $z \in S$.

To show: There exists $x \in S$ such that $z \in G\cdot x$.

Let $x = z$. Then

$z = x = 1\cdot x \in G\cdot x$.

So $S \subseteq \bigcup_{x\in S} G\cdot x$. So $S = \bigcup_{x \in S} G\cdot x$.