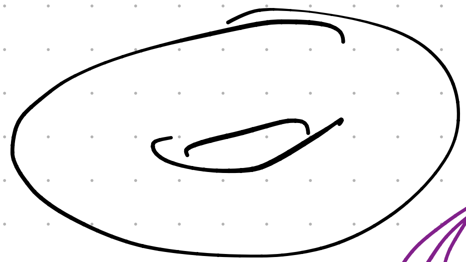
GTLA Lecture 14.08.2020



$\mathbb{C}((t))$

$\mathbb{R}$ as a $\mathbb{Q}$ vector space

$\mathbb{F}$-Vector spaces

Subspaces.

Favourite example of an $\mathbb{F}$-vector space is $\mathbb{F}^n$:

$$\mathbb{F}^n = \left\{ \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \;\middle|\; c_1, \dots, c_n \in \mathbb{F} \right\}$$

with addition

$$\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} + \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} c_1 + d_1 \\ c_2 + d_2 \\ \vdots \\ c_n + d_n \end{pmatrix}$$

and scalar multiplication

$$c \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} cc_1 \\ \vdots \\ cc_n \end{pmatrix}.$$

Linear transformations

Linear transformations are for comparing vector spaces.

Let $\mathbb{F}$ be a field.

Let V, W be $\mathbb{F}$-vector spaces

A <u>linear transformation</u> between V and W is a function

$$f : V \longrightarrow W$$

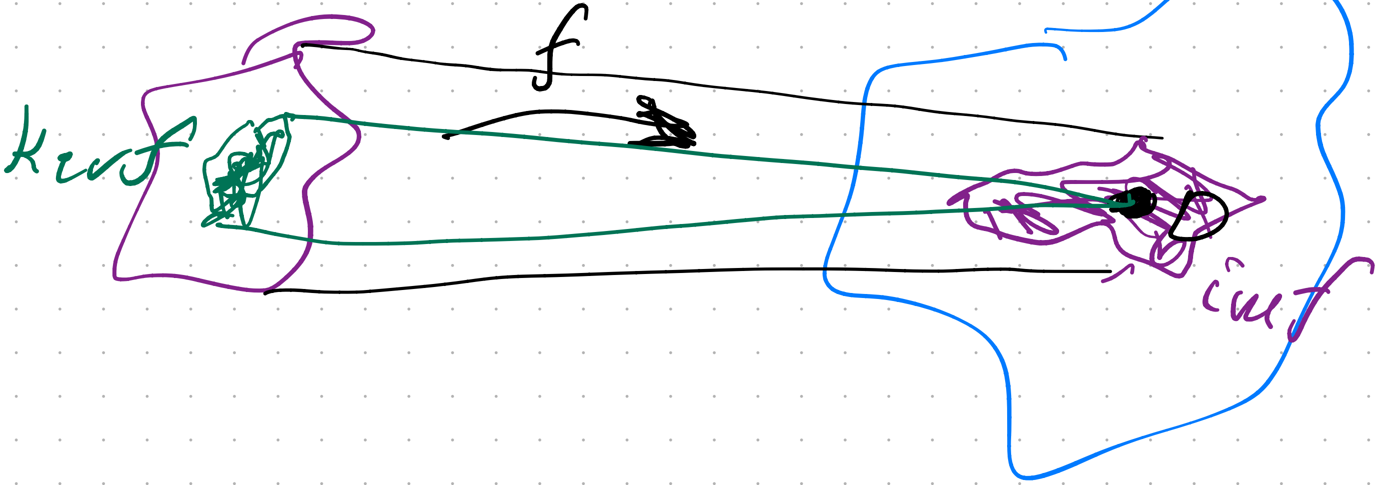
such that

(a) If $v_1, v_2 \in V$ then

$$f(v_1 + v_2) = f(v_1) + f(v_2)$$

(b) If $c \in \mathbb{F}$ and $v \in V$ then

$$f(cv) = cf(v)$$



The *image of f* is

$$\operatorname{im} f = \{ f(v) \mid v \in V \}$$

The *kernel of f*, or *nullspace of f*,

$$\ker f = \{ v \in V \mid f(v) = 0 \}.$$

Theorem

(a) $\ker f$ is a *subspace* of V

(b) $\operatorname{im} f$ is a *subspace* of W.

1^{st} way of getting matrices from vector spaces:

Matrix of linear transformation with respect to bases B and C

Let V, W be $\mathbb{F}$-vector spaces

Let $f: V \longrightarrow W$ be a linear transformation.

Let B be a basis of V

Let C be a basis of W.

The *matrix of f with respect*

to the bases B and C

is f_{CB} given by

$$f(b) = \sum_{c \in C} f_{CB}(c,b)\,c$$

for $b \in B$

$f_{CB}(c,b)$, is (c,b) entry of the matrix f_{CB}.

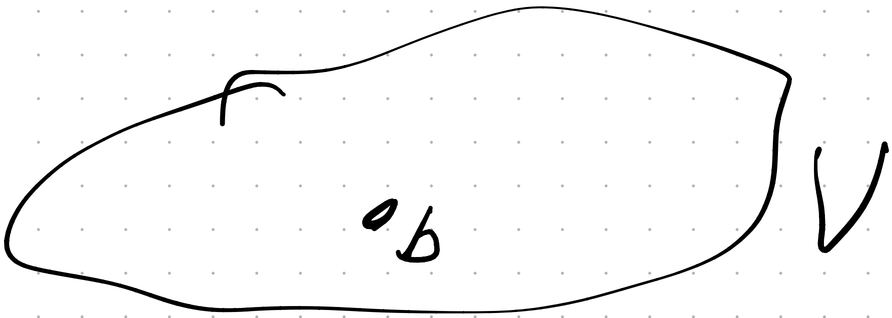
2nd way of getting matrices from vector spaces.

Let V be a vector space.
Let B and C two ~~diff~~ bases of V.
$B = \{b_1, \ldots, b_n\}$ and $C = \{c_1, \ldots, c_n\}$.

The change of basis matrix from B to C is the matrix P_{CB} given by

$$b = \sum_{c \in C} \underbrace{P_{CB}(c,b)}\, c$$

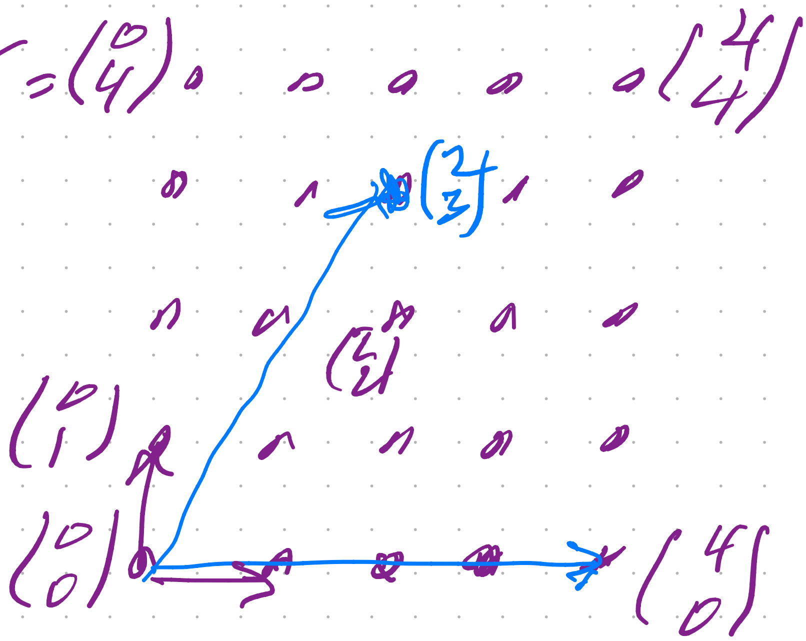
for $b \in B$



$P_{CB}(c,b)$ is the (c,b) entry of the matrix P_{CB}.

Example Let $\mathbb{F} = \mathbb{F}_5 = \{0,1,2,3,4\}$.

$V = \mathbb{F}_5^2 =$



Favorite basis "standard basis"

$B = \{b_1, b_2\}$ with $b_1 = \binom{1}{0}$, $b_2 = \binom{0}{1}$

Another basis is

$C = \{c_1, c_2\}$ with $c_1 = \binom{2}{3}$ $c_2 = \binom{4}{0}$

$$b_1 = \begin{pmatrix}1\\0\end{pmatrix} = \underline{0}\cdot\begin{pmatrix}2\\3\end{pmatrix} + \underline{4}\cdot\begin{pmatrix}4\\0\end{pmatrix} = 0\cdot c_1 + 4c_2$$

$$b_2 = \begin{pmatrix}0\\1\end{pmatrix} = 2\cdot\begin{pmatrix}2\\3\end{pmatrix} + 4\begin{pmatrix}4\\0\end{pmatrix} = 2c_1 + 4c_2$$

The change of basis matrix P_{CB} is

$$P_{CB} = \begin{pmatrix} 0 & 2 \\ 4 & 4 \end{pmatrix}$$

$$c_1 = \begin{pmatrix}2\\3\end{pmatrix} = 2\begin{pmatrix}1\\0\end{pmatrix} + 3\begin{pmatrix}0\\1\end{pmatrix} = 2b_1 + 3b_2$$

$$c_2 = \begin{pmatrix}4\\0\end{pmatrix} = 4\begin{pmatrix}1\\0\end{pmatrix} + 0\begin{pmatrix}0\\1\end{pmatrix} = 4b_1 + 0b_2$$

and so $P_{BC} = \begin{pmatrix} 2 & 4 \\ 3 & 0 \end{pmatrix}$

Then

$$P_{BC}P_{CB} = \begin{pmatrix} 2 & 4 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 4 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$b_1 = 1b_1 + 0b_2$$

$$b_2 = 0b_1 + 1b_2$$

I've set up the notation

so that

$$P_{DC} P_{CB} = P_{DB}$$

and $P_{BB} = I$.

For change of basis matrices.

Note

$P_{BC} = P_{CB}^{-1}$ since $P_{BC} P_{CB} = I$.

Let $g : V \longrightarrow W$ be a linear transformation.

Let B a basis of V

C a basis of W.

Let g_{CB} be the matrix of g with respect to B and C.

Let X a basis V

Y a basis W.

What is g_{YX}??

$$\boxed{g_{YX} = P_{YC}\, g_{CB}\, P_{BX}}$$ Useful.

If R is integral domain
(cancellation law)
then $Q = \{\frac{a}{b} \mid a, b \in R,\ b \neq 0\}$
is field

$R \subseteq Q$ $\qquad$ $R[x] \subseteq Q[x]$.

$\mathbb{Z} \subseteq \mathbb{Q}$

$\sqrt{2}x$

What is a matrix?

$A : B \times C \longrightarrow \mathbb{F}$

$A : \{1, \dots, m\} \times \{1, \dots, n\} \to \mathbb{F}$.