F.4. Proofs: Fields

Proposition F.1.1. — *If $f\colon \mathbb{K} \to \mathbb{F}$ is a field homomorphism then f is injective.*

Proof. — Proof by contrapositive.
Assume $\mathbb{K}$ and $\mathbb{F}$ are fields with additive identities $0_{\mathbb{K}}$ and $0_{\mathbb{F}}$
and multiplicative identities $1_{\mathbb{K}}$ and $1_{\mathbb{F}}$, respectively.
Assume $f\colon \mathbb{K} \to \mathbb{F}$ is a function such that

(a) If $x_1, x_2 \in \mathbb{K}$ then $f(x_1 + x_2) = f(x_1) + f(x_2)$ and
(b) If $x_1, x_2 \in \mathbb{K}$ then $f(x_1 x_2) = f(x_1)f(x_2)$.

To show: If f is not injective then f is not a field homomorphism.

Assume f is not injective.
Then there exist $x_1, x_2 \in \mathbb{K}$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$.
So $x_1 - x_2 \neq 0$.
Since $\mathbb{K}$ is a field, there exists $y \in \mathbb{K}$ such that $y(x_1 - x_2) = 1_{\mathbb{K}}$.
Then

$$\begin{aligned} f(1_{\mathbb{K}}) &= f(1_{\mathbb{K}}1_{\mathbb{K}}) = f(1_{\mathbb{K}}y(x_1 - x_2)) \\ &= f(1_{\mathbb{K}})f(y)(f(x_1) - f(x_2)) = f(1_{\mathbb{K}})f(y) \cdot 0_{\mathbb{F}} = 0_{\mathbb{F}}. \end{aligned}$$

So f is not a field homomorphism.

□

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