G.2. Group actions

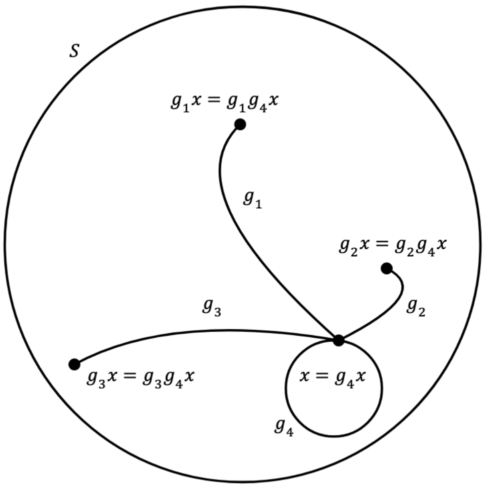
Definition G.2.1. — Let G be a group.

- A **G-set**, or **action of** G, is a set S with a function $\alpha\colon G \times S \to S$ (the convention is to write gs for $\alpha(g,s)$) such that
  (a) if $g, h \in G, s \in S$ then $g(hs) = (gh)s$,
  (b) if $s \in S$ then $1s = s$.

Examples of group actions are given below in this section and in the Exercises. One application of group actions is for proving the Sylow Theorems. See §XX.

Definition G.2.2. — Let S be a G-set and let $s \in S$.

- The **stabilizer of** s is the set
$$\mathrm{Stab}_G(s) = \{g \in G \mid gs = s\}.$$
- The **orbit of** s is the set
$$Gs = \{gs \mid g \in G\}.$$



Proposition G.2.1. — *Let S be a G-set. Let $s \in S$ and $g \in G$. Then*

(a) $\mathrm{Stab}_G(s)$ *is a subgroup of* G.

(b) $\mathrm{Stab}_G(gs) = g(\mathrm{Stab}_G(s))g^{-1}$.

The following proposition is an analogue of Proposition F.2.2 and Proposition R.1.2 and Proposition R.2.2 and Proposition G.1.2.

Proposition G.2.2. — *Let G be a group which acts on a set S. Then the orbits partition the set S.*

Corollary G.2.3. — *If G is a group acting on a set S and Gs_i denote the orbits of the action of G on S then*
$$\mathrm{Card}(S) = \sum_{\text{distinct orbits}} \mathrm{Card}(Gs_i).$$

Notes of Arun Ram aram@unimelb.edu.au, Version: 7 April 2020

It is possible to view the stabilizer G_s of an element $s \in S$ as an analogue of the kernel of a homomorphism and the orbit Gs of an element $s \in S$ as an analogue of the image of a homomorphism. One might say

	group actions $\alpha\colon G \times S \to S$	are to	group homomorphisms $f\colon G \to H$,
as	stabilizers $\mathrm{Stab}_G(s)$	are to	kernels $\ker f$,
as	orbits Gs	are to	images $\mathrm{im} f$.

From this point of view the following corollary is an analogue of Corollary G.1.4.

Proposition G.2.4. — *Let G be a group acting on a set S and let $s \in S$. If Gs is the orbit containing s and $G_s = \mathrm{Stab}_G(s)$ is the stabilizer of s then*

$$\mathrm{Card}(G/G_s) = \mathrm{Card}(Gs),$$

where G/G_s is the set of cosets of G_s in G.

Corollary G.2.5. — *Let G be a group acting on a set S. Let $s \in S$, let $\mathrm{Stab}_G(s)$ denote the stabilizer of s and let Gs denote the orbit of s. Then*

$$\mathrm{Card}(G) = \mathrm{Card}(Gs)\mathrm{Card}(\mathrm{Stab}_G(s)).$$

G.2.1. Normalizers: The conjugation action on subsets. —

Definition G.2.3. —

- Let S be a subset of a group G. The **normalizer** of S in G is the set

$$N(S) = \{x \in G \mid xSx^{-1} = S\}, \qquad \text{where } xSx^{-1} = \{xsx^{-1} \mid s \in S\}.$$

Proposition G.2.6. — *Let H be a subgroup of G and let $N(H)$ be the normalizer of H in G. Then*

(a) H is a normal subgroup of $N(H)$.

(b) If K is a subgroup of G such that $H \subseteq K \subseteq G$ and H is a normal subgroup of K then $K \subseteq N(H)$.

Proposition G.2.6 says that N_H is the largest subgroup of G such that H is normal in N_H.

Proposition G.2.7. — *Let G be a group and let $\mathcal{S}$ be the set of subsets of G. Then*
(a) *G acts on $\mathcal{S}$ by*

$$\begin{array}{rccc} \alpha\colon & G \times \mathcal{S} & \to & \mathcal{S} \\ & (g, S) & \mapsto & gSg^{-1} \end{array} \qquad \text{where } gSg^{-1} = \{gsg^{-1} \mid s \in S\}.$$

We say that G acts on $\mathcal{S}$ by ***conjugation****.*
(b) *If S is a subset of G then*

$$N_G(S) \text{ is the stabilizer of } S \quad \text{under the action of } G \text{ on } \mathcal{S}$$

by conjugation.

G.2.2. Conjugacy classes and centralizers: The conjugation action on elements. —

Definition G.2.4. — Let G be a group.

- Two elements $g_1, g_2 \in G$ are **conjugate** if there exists $h \in G$ such that $g_1 = hg_2h^{-1}$.
- Let G be a group and let $g \in G$. The **conjugacy class** of g is the set $\mathcal{C}_g$ of conjugates of g.
- Let $g \in G$. The **centralizer** of g is the set

$$Z_G(g) = \{x \in G \mid xgx^{-1} = g\}.$$

Proposition G.2.8. — *Let G be a group. Then*

(a) *G acts on G by*

$$\begin{array}{ccc} G \times G & \to & G \\ (g, s) & \mapsto & gsg^{-1}. \end{array}$$

We say that G acts on itself by conjugation.

(b) *Two elements $g_1, g_2 \in G$ are conjugate if and only if they are*

$$\text{in the same orbit} \quad \text{under the action of } G \text{ on itself}$$

by conjugation.

(c) *Let $g \in G$. The conjugacy class $\mathcal{C}_g$ of g is*

$$\text{the orbit of } g \quad \text{under the action of } G \text{ on itself}$$

by conjugation.

(d) *Let $g \in G$. The centralizer $Z_G(g)$ of g is*

$$\text{the stablilizer of } g \quad \text{under the action of } G \text{ on itself}$$

by conjugation.

Definition G.2.5. — Let G be a group.

- Let S be a subset of G. The **centralizer** of S in G is the set

$$Z_G(S) = \{x \in G \mid \text{if } s \in S \text{ then } xsx^{-1} = s\}.$$

Lemma G.2.9. — *Let $\mathrm{Stab}_G(s)$ be the stabilizer of $s \in G$ under the action of G on itself by conjugation. Then*

(a) For each subset $S \subseteq G$,

$$Z_G(S) = \bigcap_{s \in S} \mathrm{Stab}_G(s).$$

(b) $Z(G) = Z_G(G)$, where $Z(G)$ denotes the center of G.

(c) $s \in Z(G)$ if and only if $Z_G(s) = G$.

(d) $s \in Z(G)$ if and only if $\mathcal{C}_s = \{s\}$.

Proposition G.2.10. — *(The Class Equation) Let $\mathcal{C}_{g_i}$ denote the conjugacy classes in a group G. Then*

$$\mathrm{Card}(G) = \mathrm{Card}(Z(G)) + \sum_{\mathrm{Card}(\mathcal{C}_{g_i}) > 1} \mathrm{Card}(\mathcal{C}_{g_i}).$$