Calculus 2 Lecture 5
07.08.2019

(A)

Continuity

Let $p \in \mathbb{R}$. Let $f: \mathbb{R} \to \mathbb{R}$ be a function

The function f is continuous at p if

$$\lim_{x \to p} f(x) = f(p).$$

Sequences Let Y be a set.

A sequence $(y_1, y_2, \ldots)$ in Y is a function

$$\begin{aligned} \mathbb{Z}_{>0} &\to Y \\ n &\mapsto y_n. \end{aligned}$$

Series

Let $(a_1, a_2, a_3, \ldots)$ be a sequence in $\mathbb{R}$.

The series $\sum_{n=1}^{\infty} a_n$ is the sequence

$(s_1, s_2, s_3, \ldots)$ where $s_k = a_1 + a_2 + \cdots + a_{k-1} + a_k$.

Write

$$\sum_{n=1}^{\infty} a_n = \ell \quad \text{if} \quad \lim_{k \to \infty} s_k = \ell.$$

Sequence theorems

(Monomial theorem) ~~Let $n\in\mathbb{Z}_{>0}$~~. Let $x\in\mathbb{C}$.

$$\lim_{n\to\infty} x^n = \begin{cases} 0, & \text{if } |x|<1, \\ \text{diverges}, & \text{if } |x|>1, \\ 1, & \text{if } x=1, \\ \text{diverges}, & \text{if } |x|=1 \text{ and } x\neq 1. \end{cases}$$

(Geometric series theorem). Let $x\in\mathbb{C}$

$$\lim_{n\to\infty}(1+x+x^2+\cdots+x^{n-1}+x^n) = \lim_{n\to\infty}\frac{1-x^{n+1}}{1-x}$$

$$= \begin{cases} \dfrac{1}{1-x}, & \text{if } |x|<1 \\ \text{diverges}, & \text{if } |x|\geq 1 \end{cases}$$

(Ratio test) Let $(a_1, a_2, a_3, \ldots)$ be a sequence in $\mathbb{R}$.

(a) If $\lim_{n\to\infty}\dfrac{|a_{n+1}|}{|a_n|}=a$ exists and

$a<1$ then $\sum_{n=1}^{\infty}|a_n|$ converges.

(b) If $\lim_{n\to\infty}\dfrac{|a_{n+1}|}{|a_n|}=a$ exists and

$a>1$ then $\sum_{n=1}^{\infty}|a_n|$ diverges.

<u>Example 1.23</u> Evaluate $\lim_{n\to\infty}\left(\left(\frac{n-2}{n}\right)^n + \frac{4n^2}{3^n}\right)$

<u>Solution:</u> Use the identities

$$\lim_{n\to\infty}\left(1+\frac{a}{n}\right) = e^a \quad \text{and} \quad \lim_{n\to\infty}\frac{n^p}{a^n} = 0$$

to conclude that

$$\lim_{n\to\infty}\left(\left(\frac{n-2}{n}\right)^n + \frac{4n^2}{3^n}\right) = \lim_{n\to\infty}\left(\left(1+\frac{-2}{n}\right)^n + 4\,\frac{n^2}{3^n}\right)$$

$$= e^{-2} + 4\cdot 0 = e^{-2}.$$

<u>Example 1.24</u> Find the limit of the sequence

$$a_n = \frac{3^n+2}{4^n+2^n} \quad \text{for } n \in \mathbb{Z}_{>0}.$$

<u>Solution</u> $$\lim_{n\to\infty}\frac{3^n+2}{4^n+2^n} = \lim_{n\to\infty}\frac{(3^n+2)\frac{1}{4^n}}{(4^n+2^n)\frac{1}{4^n}}$$

$$= \lim_{n\to\infty}\frac{\left(\frac{3}{4}\right)^n + \frac{2}{4^n}}{1+\left(\frac{2}{4}\right)^n} = \lim_{n\to\infty}\frac{\left(\frac{3}{4}\right)^n + \frac{2}{4^n}}{1+\frac{1}{2^n}} = \lim_{n\to\infty}\left(\frac{3}{4}\right)^n = 0$$

since $\frac{3}{4} < 1$.

Example 1.25 Prove that $\lim_{n\to\infty} \frac{\log n}{n^p} = 0$ if $p \in \mathbb{R}_{>0}$.

Solution: Let $p \in \mathbb{R}_{>0}$. Substituting $n = e^y$

$$\lim_{n\to\infty} \frac{\log n}{n^p} = \lim_{y\to\infty} \frac{\log e^y}{e^{py}} = \lim_{y\to\infty} \frac{y}{e^{py}}$$

$$= \lim_{y\to\infty} \frac{y}{1+py+\frac{1}{2}p^2y^2+\cdots} = \lim_{y\to\infty} \frac{1}{\frac{1}{y}+p+\frac{1}{2}p^2y+\cdots}$$

$$= 0,$$

since $\lim_{y\to\infty} \frac{1}{y}+p+\frac{1}{2}p^2y+\cdots = \infty$.

Example 1.26 Evaluate $\lim_{n\to\infty} (\log(3n-2)-\log n)$

Solution: $\lim_{n\to\infty} (\log(3n-2)-\log n) = \lim_{n\to\infty} \log\left(\frac{3n-2}{n}\right)$

$$= \lim_{n\to\infty} \log\left(3-\frac{2}{n}\right) = \log 3.$$

Example 1.27 Evaluate $\lim_{n\to\infty} \frac{1+\sin^2\left(\frac{n\pi}{3}\right)}{\sqrt{n}}$

Solution: $\lim_{n\to\infty} \frac{1+\sin^2\left(\frac{n\pi}{3}\right)}{\sqrt{n}} \leq \lim_{n\to\infty} \frac{2}{n^{\frac{1}{2}}} = 0$

Since $0 \leq \frac{1+\sin^2\left(\frac{n\pi}{3}\right)}{\sqrt{n}}$ then $0 \leq \lim_{n\to\infty} \frac{1+\sin^2\left(\frac{n\pi}{3}\right)}{\sqrt{n}} \leq 0$.

So $\lim_{n\to\infty} \frac{1+\sin^2\left(\frac{n\pi}{3}\right)}{\sqrt{n}} = 0$.

Example 1.28 Find the sum of $a_n = \left(\frac{1}{2}\right)^n$ for $n \in \mathbb{Z}_{>0}$.

Solution: $s_1 = a_1 = \frac{1}{2}$

$s_2 = a_1 + a_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$s_3 = a_1 + a_2 + a_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$

$s_4 = a_1 + a_2 + a_3 + a_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$

$\vdots$

So $$\sum_{n \in \mathbb{Z}_{>0}} \left(\frac{1}{2}\right)^n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{n-1}{n} = \lim_{n\to\infty} 1 - \frac{1}{n} = 1.$$

Example 1.29 What does the series
$$\sum_{n \in \mathbb{Z}_{\geq 0}} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots$$ converge to?

Solution: $$\sum_{n \in \mathbb{Z}_{\geq 0}} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2.$$

Example 1.30 Does the series $\sum_{n=1}^{\infty} \frac{n+1}{n}$ converge?

Solution Since $\frac{n+1}{n} = 1 + \frac{1}{n} > 1$ for $n \in \mathbb{Z}_{>0}$ then
$$\sum_{n=1}^{\infty} \frac{n+1}{n} > \sum_{n=1}^{\infty} 1.$$

Since $\sum_{n=1}^{\infty} 1$ is infinite and does not converge then $\sum_{n=1}^{\infty} \frac{n+1}{n}$ is infinite and does not converg