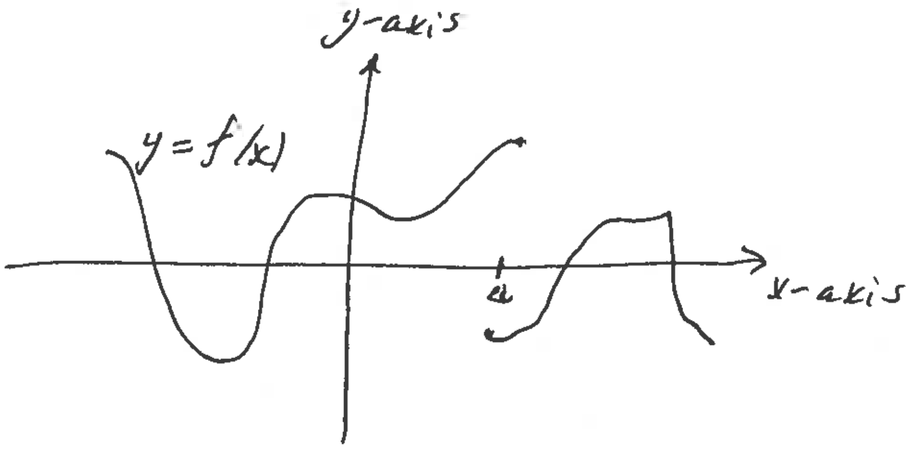
Calculus 2 Lecture 3 02.08.2019
A. Ram

A

A function $f(x)$ is continuous at a if it doesn't jump at $x=a$, i.e.

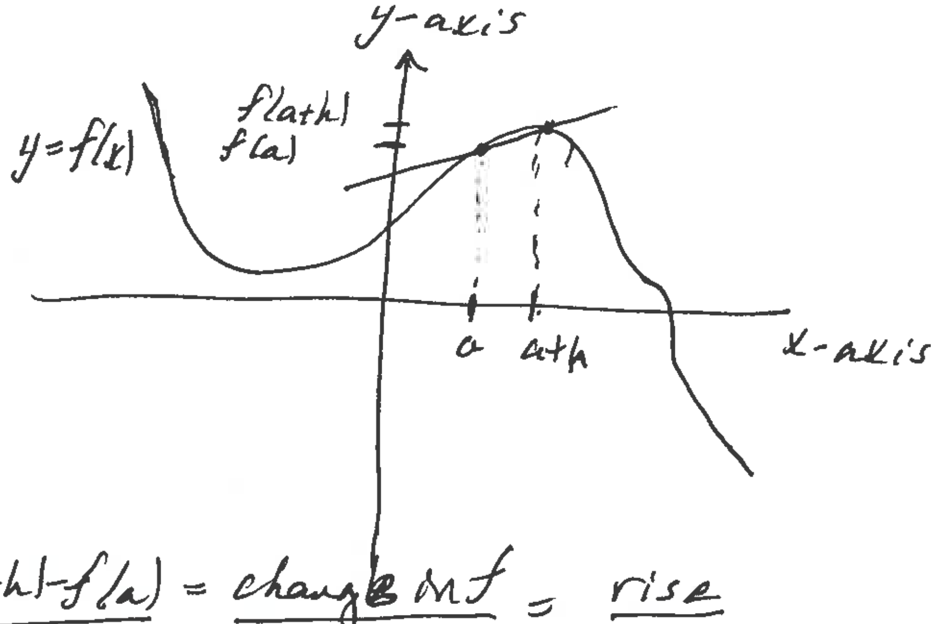
$$\lim_{x\to a} f(x) = f(a)$$



Not continuous at $x=a$

The differential of f at $x=a$ is

$$\left.\frac{df}{dx}\right|_{x=a} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$



$$\frac{f(a+h)-f(a)}{h} = \frac{\text{change in } f}{\text{change in } x} = \frac{\text{rise}}{\text{run}}$$

= slope of line connecting $(a, f(a))$ and $(a+h, f(a+h))$, Continuity Theorems

Let $f: \mathbb{R} \to \mathbb{R}$ be a function.

The function f is continuous if f satisfies:

if $p \in \mathbb{R}$ then f is continuous at p.

(Addition theorem). Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be functions and let $a \in \mathbb{R}$.

If f is continuous at a and

g is continuous at a

then $f+g$ is continuous at a.

(Scalar multiplication theorem) Let $f: \mathbb{R} \to \mathbb{R}$ be a function and let $a \in \mathbb{R}$. Let $c \in \mathbb{R}$

If f is continuous at a ~~and~~ then

cf is continuous at a

(Multiplication theorem) Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be functions and let $a \in \mathbb{R}$.

If f is continuous at a and

g is continuous at a

then

fg is continuous at a.

(Composition of functions theorem)

Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be functions and let $a \in \mathbb{R}$.

If f is continuous at a and
g is continuous at $f(a)$
then $g \circ f$ is continuous at a.

(Power theorem) Let $n \in \mathbb{Z}_{>0}$ and let $a \in \mathbb{R}$.

(1) x^n is continuous at a.

(2) If $a \neq 0$ then x^{-n} is continuous at a.

(3) x^0 is continuous at a.

(Exponential theorem) Let $a \in \mathbb{R}$.

(1) e^x is continuous at a.

(2) $\sin x$ is continuous at a

(3) $\cos x$ is continuous at a

(4) $\sinh x$ is continuous at a

(5) $\cosh x$ is continuous at a

(Division theorem) Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be functions and let $a \in \mathbb{R}$.

If f is continuous at a and
g is continuous at a and
$g(a) \neq 0$

then $\frac{f}{g}$ is continuous at a.

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D. Rawe

Example 1.14 Let $f(x) = \dfrac{\log x + \sin x}{\sqrt{x^2-1}}$.

For which values of x is f continuous?

Solution $\sin x$ is continuous for $x \in \mathbb{R}$

$\log x$ is continuous for $x \in \mathbb{R}_{>0}$

$\sqrt{x^2-1}$ is continuous for $x^2 \in \mathbb{R}_{\geq 1}$.

$\sqrt{x^2-1} \neq 0$ if $x^2 \in \mathbb{R}_{>1}$.

So $f(x) = (\log x + \sin x)\dfrac{1}{\sqrt{x^2-1}}$ is continuous for

$x^2 \in \mathbb{R}_{>1}$.

Since $\{x \in \mathbb{R} / x^2 \in \mathbb{R}_{>1}\} = \{x \in \mathbb{R} / \mathbb{R}_{>1}\} \sqcup \{x \in \mathbb{R} / x \in \mathbb{R}_{<-1}\}$

then

$f(x) = (\log x + \sin x)\dfrac{1}{\sqrt{x^2-1}}$ is continuous for

$x \in \mathbb{R}_{<-1} \sqcup \mathbb{R}_{>1}$.

Example 1.15 Let $c \in \mathbb{R}$ and

$$f(x) = \begin{cases} x^3 - cx + 8, & \text{if } x \in \mathbb{R}_{\le 1} \\ x^2 + 2cx + 2, & \text{if } x \in \mathbb{R}_{>1} \end{cases}$$

For which values of c is f continuous?

Solution: Let $c \in \mathbb{R}$.

If $x \in \mathbb{R}_{<1}$ then $f(x) = x^3 - cx + 8$ and

$x^3 - cx + 8$ is a polynomial and is continuous.

If $a \in \mathbb{R}_{>1}$ then $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} x^2 + 2cx + 2$

$= \cancel{\lim}\ a^2 + 2ca + 2 = f(a)$

and so $f(x)$ is continuous for $a \in \mathbb{R}_{>1}$.

$$\lim_{\substack{x \to 1 \\ x \in \mathbb{R}_{<1}}} f(x) = \lim_{\substack{x \to 1 \\ x \in \mathbb{R}_{<1}}} x^3 - cx + 8 = 1^3 - c \cdot 1 + 8 = 9 - c$$

$$\lim_{\substack{x \to 1 \\ x \in \mathbb{R}_{>1}}} f(x) = \lim_{\substack{x \to 1 \\ x \in \mathbb{R}_{>1}}} x^2 + 2cx + 2 = 1^2 + 2c \cdot 1 + 2 = 3 + 2c.$$

So $\lim\limits_{x \to 1} f(x)$ exists only if $9 - c = 3 + 2c$

So $\lim\limits_{x \to 1} f(x)$ exists if $3c = 6$.

If $c = 2$ then $\lim\limits_{x \to 1} f(x) = 9 - 2 = 7 = 1^3 - c \cdot 1 + 8 = \overset{f(1)}{\cancel{9-2=7}}$

and f is continuous at $x = 1$.

If $c \neq 2$ then $\lim\limits_{x \to 1} f(x)$ does not exist and f is not continuous at $x = 1$.

A. Ram

Calculus 2

Example 1.16 Evaluate $\lim_{x\to\infty} \sin(e^{-x})$.

Solution Since $\lim_{x\to\infty} e^{-x} = \lim_{x\to-\infty} e^{x} = 0$ then
$\lim_{x\to\infty} \sin(e^{-x}) = \lim_{y\to 0} \sin y = 0$.

Example 1.20 Prove that $\lim_{x\to 1} 2x = 2$.

Prove To show: If $\varepsilon \in \mathbb{E}$ then there exists $\delta \in \mathbb{E}$ such that
if $d(x,1) < \delta$ then $d(f(x), 2) < \varepsilon$.

Assume $\varepsilon = 10^{-k}$.
Let $\delta = 10^{-(k+1)}$.
To show: If $d(x,1) < \delta$ then $d(f(x), 2) < \varepsilon$.
Assume $d(x,1) < \delta$.
So $|x-1| < 10^{-(k+1)} = \frac{1}{10} 10^{-k}$
To show: $d(f(x), 2) < \varepsilon$.

$$d(f(x),2) = |f(x)-2| = |2x-2| = 2|x-1|$$
$$< 2\tfrac{1}{10} 10^{-k} < \tfrac{1}{5}\varepsilon < \varepsilon.$$